

Continuity of solutions of a class of fractional equations

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Abstract

In practice many problems related to space/time fractional equations depend on fractional parameters. But these fractional parameters are not known a priori in modelling problems. Hence continuity of the solutions with respect to these parameters is important for modelling purposes. In this paper we will study the continuity of the solutions of a class of equations including the Abel equations of the first and second kind, and time fractional diffusion type equations. We consider continuity with respect to the fractional parameters as well as the initial value.

Keywords: Space-time-fractional partial differential equations; Caputo derivatives; Abel equation of the first kind; Abel equation of the second kind; time fractional diffusion in Banach spaces

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1 Introduction

Diffusion is one of the most important transport mechanisms found in nature. At a microscopic level, the diffusion is caused by random motion of individual particles. In fact, let $x(t)$, $t > 0$, be the displacement of a particle at time t . From the theory of random walk (see, e.g., [24]) the mean squared displacement grows as

$$\mathbb{E}(x^2(t)) = \langle x^2(t) \rangle \sim t^\alpha,$$

where the constant $\alpha > 0$ can be called the *order of diffusion*. Classical model of diffusion corresponds to the case $\alpha = 1$. In this case, the random walk limit is modeled by a Brownian motion with the corresponding Laplacian operator. The corresponding model for $\alpha \neq 1$ is called *anomalous diffusion*. A growing number of studies on the diffusion phenomena have shown the prevalence of anomalous diffusion in which the mean square variance grows faster (in the case of superdiffusion, i.e., $\alpha > 1$) or slower (in the case of subdiffusion, i.e., $0 \leq \alpha < 1$) than the one for diffusion processes. From the mathematical point of view, to model the anomalous diffusion, we use fractional derivatives of t (of order $\alpha \notin \mathbb{Z}$) instead of the classical derivatives. Experiments showed that the fractional derivative models lead to explaining and understanding complex systems better with the use of anomalous diffusion processes. The fractional models, dated back to the 19th century, has never ceased to inspire scientists and engineers to investigate this area of research deeper. Nowadays, anomalous diffusion became 'normal' in spatially disordered systems, porous media, fractal media, viscoelastic materials [9], [14], [26], pollution transport, turbulent fluids and plasmas [3],[4], [18], biological media with traps, binding sites or macro-molecular crowding [11],[12], stock price movements [29], [31].

The parameter α is an important constant in the model of anomalous diffusion. For example, a simple anomalous diffusion can be described macroscopically by the fractional diffusion equation

$$\partial_t^\alpha u(x, t) = \Delta u(x, t), \quad x \in \Omega \subset \mathbb{R}^k \quad (1.1)$$

where $u(x, t)$ is the probability of finding a particle at spatial point x and time t and that $\partial_t^\alpha u = \frac{\partial^\alpha u}{\partial t^\alpha}$ is the Caputo fractional derivative of the function u . The parameter α can only be determined experimentally. Therefore, we cannot obtain the exact value of α and we often have a known sequence α_n , called the perturbed fractional parameter, satisfying $\alpha_n \xrightarrow{n \rightarrow \infty} \alpha$ in an appropriate sense. In fact, if the sequence (α_n) is deterministic, we have $\lim_{n \rightarrow \infty} |\alpha_n - \alpha| = 0$ and if the sequence is random, we can assume that $\lim_{n \rightarrow \infty} \mathbb{E}|\alpha_n - \alpha|^\lambda = 0$. Hence, in these cases, we only find perturbed solutions of (1.1). The question is whether the perturbed solutions are stable with respect to the parameters α_n . It is surprising to see that the inexact nature of the parameter α is not investigated in the literature of fractional calculus.

For fixed α the solutions of (1.1) have been investigated recently. Existence and uniqueness of the solutions as well as numerical methods for solving forward fractional equations are well developed. One can find papers devoted to the Abel (and generalized Abel) equations in Gorenflo-Vessella [16] and references therein. Fractional diffusion equations in Banach spaces were studied by Bazhlekova [1], Clément [5],[6]. Fractional diffusion equations in Hilbert spaces were considered

by Chen et al. [8], Li and Xu [20], Meerschaert et al. [23], Sakamoto and Yamamoto [30], Zacher [34], and many others. There is also an ever-growing literature of research on the fractional inverse problem with the exact fractional order. For example, the backward problems for fractional diffusion processes, which are ill posed, corresponding to the irreversibility of time, have a rich literature: see, for example, [7], [22], [30], [35].

For $\alpha > 0$, the common problem of interest is finding the solution of the general problem of the form given by the equation $A_\alpha u_\alpha = f$ where the function f is given, u_α is unknown and A_α is a kind of fractional operator; for example, $A_\alpha = \partial_t^\alpha$ is the Caputo fractional derivative. If the inexact nature of the fractional parameter α is present then the continuity of the solution with respect to the parameter α has to be considered in numerical considerations. Suppose that a sequence (α_n) satisfies that $\alpha_n \rightarrow \alpha$ as $n \rightarrow \infty$. Since α is unknown, in real life applications, we can only compute an approximation u_{α_n} of u_α . This situation raises the following natural questions:

- (a) Does $u_{\alpha_n} \rightarrow u_\alpha$ in an appropriate sense as $n \rightarrow \infty$?
- (b) If $u_{\alpha_n} \not\rightarrow u_\alpha$ then, is there another way to recover the convergence?

These questions are related with the continuity of solution of fractional problems with respect to the fractional parameter α . For brevity, we shall call the investigation of these questions by *the parameter-continuity problem*. Papers investigating these two questions for the fractional problem is very rare. To the best of our knowledge, there are a few papers related to these questions. The paper [17] was devoted to the problem of determination of the parameter in a fractional diffusion equation. These authors considered the fractional Cauchy problem in a domain $\Omega \subset \mathbb{R}^k$, and determine the parameter α from the observed data measured at a point inside Ω . In [1, Theorem 4.2], using the semigroup language, the author gave a formula which described the relation between the solutions of the fractional Cauchy problem for two parameters $\alpha, \beta \in (0, 1]$. The papers [2], [10], [21], [28] dealt with the problem of simultaneously identifying the fractional parameter and the space-dependent diffusion coefficient from boundary measurements. In these papers, some results on the Lipschitz continuity of solutions with respect to the fractional parameter were proved. In [21, Proposition 1], the authors considered the problem of finding a function $u = u_{\gamma,D}(x, t)$ ($0 < \gamma < 1$) satisfying

$$\frac{\partial^\gamma u}{\partial t^\gamma} = \frac{\partial}{\partial x} \left(D(x) \frac{\partial u}{\partial x} \right), \quad x \in (0, 1), t \in (0, T),$$

subject to the Neumann boundary condition $u_x(0, t) = u_x(1, t) = 0$ and the initial condition $u(x, 0) = f(x)$. They proved the Lipschitz continuity

$$\|u_{\gamma,D}(0, t) - u_{\tilde{\gamma},\tilde{D}}(0, t)\|_{L^2(0,T)} \leq C(|\gamma - \tilde{\gamma}| + \|D - \tilde{D}\|_{C[0,1]})$$

where $\|D - \tilde{D}\|_{C[0,1]} := \sup_{0 \leq x \leq 1} |D(x) - \tilde{D}(x)|$.

The latter result in [21] is a kind of parameter-continuity result. To the best of our knowledge, there are no papers in the literature that consider these problems mentioned above systematically.

In this paper, inspired by the above discussion we study systematically the continuity of the solution of equations similar to (1.1) with respect to the parameter α . Our methods are different than the method used in [21], and we obtain continuity of the solutions of abstract time fractional equation in Banach space setting with respect to the time fractional parameter as well as the initial function.

To give a sense of our results, we mention a particular case of our results in the simplest case of time fractional diffusion in the interval $(0, L)$. The equation

$$\begin{aligned}\frac{\partial^2 v(x)}{\partial x^2} &= -\lambda v(x), \quad x \in (0, L), \\ v(0) &= 0 = v(L),\end{aligned}$$

is solved by a sequence of eigenvalues $\lambda_n = (n\pi/L)^2$ and eigenfunctions $\phi_n(x) = \sqrt{\frac{2}{L}} \sin(n\pi x/L)$. It is well known that the set of functions $\{\sqrt{\frac{2}{\pi}} \sin(n\pi x/L) : n \in \mathbb{N}\}$ is an orthonormal basis of $L^2(0, L)$. Let $\alpha \in (0, 1)$. Then by separation of variables the solution of time fractional heat equation in $(0, L)$

$$\begin{aligned}\partial_t^\alpha u(x, t) &= \frac{\partial^2 u(x, t)}{\partial x^2}, \quad x \in (0, L), \quad t > 0, \\ u(x, 0) &= \theta(x), \quad x \in (0, L), \\ u(0, t) &= 0 = u(L, t)\end{aligned}\tag{1.2}$$

is given by

$$u(x, t) = \sum_{n=1}^{\infty} \theta_n E_{\alpha, 1}(-(n\pi/L)^2 t^\alpha) \sin(n\pi x/L)$$

where $\theta_n = \int_0^L \theta(x) \sin(n\pi x/L) dx$, and $E_{\alpha, 1}(-(n\pi/L)^2 t^\alpha)$ is the Mittag-Leffler function defined below in equation (2.1). To emphasize the dependence of the solution of equation (1.2) to the initial value and α , we write $u(x, t) = u_{\theta, \alpha}(x, t)$. A particular case of our Theorem 4.3 shows the following continuity properties of the solution of equation (1.2):

Let $\alpha, \alpha' \in (0, 1)$, $\theta, \theta' \in H^1$. (H^1 is domain of Laplacian in the interval $(0, L)$ defined below in section 4.2)

(i) If $\theta' \rightarrow \theta$ in H^1 , $\alpha' \rightarrow \alpha$ then

$$\|u_{\alpha', \theta'} - u_{\alpha, \theta}\|_{L^2(0, L)} \rightarrow 0.$$

(ii) If $\theta, \theta' \in H^1$, $1 > \rho \geq 0$, $\alpha' \in [\alpha_0, \alpha_1]$, then there exists a constant $C = C(\alpha_0, \alpha_1, \rho)$ such that

$$\|u_{\alpha', \theta'}(\cdot, t) - u_{\alpha, \theta}(\cdot, t)\|_{H^\rho}^2 \leq 2\|\theta' - \theta\|_{H^1}^2 + C\|\theta\|_{H^1}^2(|\alpha' - \alpha|)^{2\gamma},$$

where $\gamma = \min\{1, (1 - \rho)/\beta_1\}$, and H^ρ is defined in Section 4.2.

We will prove continuity properties of fractional differential equations of Abel type as well as abstract time fractional Cauchy problems in Banach space and Hilbert space settings with external force terms with respect to various parameters including the time fractional derivative parameter $\alpha \in (0, 1)$.

Next, we give an outline of the paper. In Section 2, we will give some definitions and basic properties of fractional derivatives. Using the definition stated in Zygmund [36, page 134] we will define the fractional derivative in a general form in Banach spaces. We will also give some properties of Mittag-Leffler functions that are used frequently in fractional problems in this paper. In Section 3, we investigate the first question of continuity with respect to the parameter α . In particular, we will consider the forward fractional problems of the generalized Abel equations. In section 4 we will consider abstract fractional diffusion equations in the Banach space setting,

and Hilbert space setting. In sections 3 and 4, we will show that the parameter-continuity is of Lipschitz continuity type in most of the cases we consider. In Section 5, we investigate the second question. For many inverse problems $A_\alpha u_\alpha = f$, we have $u_{\alpha_n} \not\rightarrow u_\alpha$ as $n \rightarrow \infty$. This prevents one to approximate solutions by numerical methods. We shall define a new concept of regularization (of the family of operators A_α) for this case and derive a regularization operator for the fractional backward problems.

2 Fractional derivatives and the Mittag-Leffler functions

We need some notations and properties in order to state our problems precisely. We barrow some definitions stated in Zygmund [36], page 134-136, to define fractional derivatives. Put

$$k_\alpha(t) = \frac{1}{\Gamma(\alpha)} t^{\alpha-1} \quad \text{for } t > 0, \alpha > 0,$$

here $\Gamma(\cdot)$ is the standard Gamma function defined by

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt, \quad \Re(z) > 0.$$

Let \mathbb{X} be a Banach space and $f \in L^1(0, T, \mathbb{X})$, we denote the Riemann-Liouville fractional integral operator (see, e.g., Gorenflo-Vessella [16]) by

$$J^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds = k_\alpha * f(t).$$

For $u \in L^1(0, T, \mathbb{X})$, if $J^{1-\alpha}u$ is absolutely continuous then we define the Riemann-Liouville fractional derivative of order $\alpha \in (0, 1)$ of u by

$$D_t^\alpha u(t) := \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-s)^{-\alpha} u(s) ds.$$

If u is absolutely continuous and differentiable a.e. then we define the (left-sided) Caputo fractional derivative of order α by

$$\partial_t^\alpha u = \frac{\partial^\alpha u}{\partial t^\alpha} := \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} u'(s) ds.$$

Let $\eta_j \in (0, 1]$, $j = 1, \dots, m$, $\sigma_j = \sum_{\ell=1}^j \eta_\ell$. We introduce the notation for the Miller-Ross sequential derivatives (see [27], page 108])

$$\begin{aligned} \mathcal{D}_t^{\sigma_m} &= D_t^{\eta_m} D_t^{\eta_{m-1}} \dots D_t^{\eta_1}, \\ \mathcal{D}_t^{\sigma_{m-1}} &= D_t^{\eta_{m-1}} D_t^{\eta_{m-2}} \dots D_t^{\eta_1} \end{aligned}$$

with

$$\sigma_m = \sum_{j=1}^m \eta_j, \quad 0 < \eta_j \leq 1, \quad j = 1, \dots, m.$$

From the definition of the Riemann-Liouville fractional derivative, we have the following lemma which collects some well-known facts about fractional derivative.

Lemma 2.1. (a) Let $0 < \alpha < 1$ and $u \in L^1(0, T; \mathbb{X})$. If there exist $f \in L^1(0, T; \mathbb{X})$ such that $u = J^\alpha f$ then the function u has the fractional derivative $D_t^\alpha u = f$.

(b) If $D_t^\alpha u \in L^p(0, T; \mathbb{X})$ with $1 \leq p < \alpha^{-1}$ then $u \in L^q(0, T; \mathbb{X})$ with $q \in [1, \frac{p}{1-\alpha p})$.

(c) If $D_t^\alpha u \in L^p(0, T; \mathbb{X})$ with $p = \alpha^{-1}$ then $u \in L^q(0, T; \mathbb{X})$ with $q \geq 1$.

(d) If $D_t^\alpha u \in L^p(0, T; \mathbb{X})$ with $p > \alpha^{-1}$ then u is Hölder continuous with exponent $\theta = \alpha - p^{-1}$ and $u(0) = 0$.

(e) For $c \in \mathbb{X}$ we have

$$D_t^\alpha c = \frac{c}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-s)^{-\alpha} ds = \frac{ct^{-\alpha}}{\Gamma(1-\alpha)}.$$

(f) For $u \in L^1(0, T; \mathbb{X})$ and $0 < \alpha \leq \beta < 1$, we have

$$D_t^\alpha J^\beta u = J^{\beta-\alpha} u.$$

(g) For an absolutely continuous u we have

$$D_t^\alpha (u(t) - u(0)) = \partial_t^\alpha u(t).$$

Proof. We first prove a). We have $J^{1-\alpha} u = J^{1-\alpha} J^\alpha f = J^1 f$. It follows that $f = \frac{d}{dt} J^{1-\alpha} u = D_t^\alpha u$. Proofs of the results (b), (c), (d), (e) can be found in Zygmund [36], pages 134-136. The proof of the results (f) and (g) can be seen in [16, Chap. 6, page 98], . \square

In this paper we consider the Mittag-Leffler functions defined by

$$E_{\alpha, \beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha + \beta)}, \quad z \in \mathbb{C}. \quad (2.1)$$

The Mittag-Leffler function is a two-parameter family of entire functions of z of order α^{-1} and type 1 [27, Chap.1]. The exponential function is a particular case of the Mittag-Leffler function, namely $E_{1,1}(z) = e^z$. Two important functions derived from this family are $E_{\alpha,1}(-\lambda t^\alpha)$ and $t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^\alpha)$, which occur in the solution operators for the initial value problem (4.7) and the nonhomogeneous problem (4.8), respectively.

We will use the next lemma for the derivatives of some contour integrals.

Lemma 2.2. Let $\alpha_0, \alpha_1, \beta_0 \in \mathbb{R}$ satisfy $0 < \alpha_0 < \alpha_1 < 2$, $\alpha_1 < 2\alpha_0$. Let $\varphi \in (\frac{\pi\alpha_1}{2}, \pi\alpha_0)$, and put

$$g_0(r) := \exp \left(r^{1/\alpha_0} \cos \left(\frac{\varphi}{\alpha_1} \right) \right).$$

Then the function $r^\mu |\ln(r)|^\nu g_0(r)$ is in $L^1(\rho, \infty)$ for every $\mu, \nu \in \mathbb{R}, \nu > 0, \rho > 0$ and

$$\left| \exp(r^{1/\alpha} e^{\pm i\varphi/\alpha}) \right| \leq g_0(r).$$

Proof. We note that

$$\left| \exp(r^{1/\alpha} e^{\pm i\varphi/\alpha}) \right| = \exp\left(r^{1/\alpha} \cos\left(\frac{\varphi}{\alpha}\right)\right).$$

From the choice of φ , we obtain

$$\frac{\pi}{2} < \frac{\varphi}{\alpha_1} < \frac{\varphi}{\alpha} < \frac{\varphi}{\alpha_0} < \pi.$$

Since the function $\cos x$ is decreasing in $(\pi/2, \pi)$, we obtain

$$-1 < \cos\left(\frac{\varphi}{\alpha_0}\right) \leq \cos\left(\frac{\varphi}{\alpha}\right) \leq \cos\left(\frac{\varphi}{\alpha_1}\right) < 0$$

for every $\alpha \in [\alpha_0, \alpha_1]$. Therefore

$$\left| \exp(r^{1/\alpha} e^{\pm i\varphi/\alpha}) \right| = \exp\left(r^{1/\alpha} \cos\left(\frac{\varphi}{\alpha}\right)\right) \leq g_0(r).$$

Since $\cos\left(\frac{\varphi}{\alpha_1}\right) < 0$, the function $r^\mu g_0(r) |\ln r|^\nu$ is Lebesgue integrable on $[\rho, \infty)$ for every $\mu, \nu \in \mathbb{R}, \nu > 0$. This completes the proof of our lemma. \square

In the following lemma, we show some inequalities which hold uniformly for all the fractional parameter α in an interval $[\alpha_0, \alpha_1]$. A few of these inequalities will not to be used in the present paper. However, they also are presented here since they can be applied for future papers. This lemma also establishes some properties of some generalized Gronwall type inequalities.

Lemma 2.3. *Let $\alpha > 0, \beta > 0$. Then $E_{\alpha,\beta}(z)$ is differentiable with respect to α, β, z . Moreover, assume that $a_0, b_0, M, \alpha_0, \alpha_1, \beta_0, \beta_1 \in \mathbb{R}$ satisfy $a_0, b_0, M > 0, 0 < \alpha_0 < \alpha_1 < 2, \alpha_1 < 2\alpha_0, \beta_0 < \beta_1$.*

(a) *For $\alpha \geq a_0, \beta \geq b_0$, there exists a constant $C_E = C_E(a_0, b_0)$ such that*

$$\begin{aligned} |E_{\alpha,\beta}(z)| &\leq C_E E_{a_0,b_0}(M) \quad \text{for all } z \in \mathbb{C}, 0 \leq |z| \leq M, \\ 0 \leq E_{\alpha,\beta}(z) &\leq C_E E_{a_0,b_0}(M) \quad \text{for all } z \in \mathbb{R}, 0 \leq z \leq M, \end{aligned}$$

and, for $z_0 \in \mathbb{R}, \alpha_0 \leq \alpha \leq \alpha_1$ there exists a constant $C = C(z_0, \alpha_0, \alpha_1, \beta_0) > 0$ such that

$$\begin{aligned} |E_{\alpha,\beta}(z)| + \left| \frac{\partial E_{\alpha,\beta}}{\partial \alpha}(z) \right| + \left| \frac{\partial E_{\alpha,\beta}}{\partial \beta}(z) \right| &\leq \frac{C}{1 + |z|}, \\ \left| \frac{\partial E_{\alpha,\beta}}{\partial z}(z) \right| &\leq \frac{C}{(1 + |z|)^2} \quad \text{for all } z < z_0. \end{aligned}$$

We also have the Lipschitz continuity

$$|E_{\alpha,\beta}(z_1) - E_{\alpha,\beta}(z_2)| \leq C|z_1 - z_2|$$

for every z_1, z_2 in $(-\infty, z_0]$.

- (b) Let $z_1 \in \mathbb{R}, z_1 > 0$ and put $\phi_0(\alpha, \beta, z) = \frac{1}{\alpha} z^{(1-\beta)\alpha} e^{z^{\frac{1}{\alpha}}}$. For $z \geq z_1 > 0$, there exists a constant $C = C(z_1, \alpha_0, \alpha_1, \beta_0) > 0$ such that

$$\begin{aligned} |E_{\alpha, \beta}(z) - \phi_0(\alpha, \beta, z)| &\leq \frac{C}{1 + |z|}, \\ \left| \frac{\partial E_{\alpha, \beta}}{\partial \alpha}(z) - \frac{\partial \phi_0}{\partial \alpha}(\alpha, \beta, z) \right| &\leq \frac{C}{1 + |z|}, \\ \left| \frac{\partial E_{\alpha, \beta}}{\partial \beta}(z) - \frac{\partial \phi_0}{\partial \beta}(\alpha, \beta, z) \right| &\leq \frac{C}{1 + |z|}. \end{aligned}$$

- (c) For $z \geq z_1 > 0, \alpha_0 \leq \alpha \leq \alpha_1, \beta_0 \leq \beta \leq \beta_1$, there exists constants $C^-, C^+ > 0$ depending only on $z_1, \alpha_0, \alpha_1, \beta_0, \beta_1$ such that

$$C^- \phi_0(\alpha, \beta, z) \leq E_{\alpha, \beta}(z) \leq C^+ \phi_0(\alpha, \beta, z).$$

Epecially, for $\beta = 1, z \geq 0$, we have

$$\frac{C^-}{\alpha} e^{z^{\frac{1}{\alpha}}} \leq E_{\alpha, 1}(z) \leq \frac{C^+}{\alpha} e^{z^{\frac{1}{\alpha}}}.$$

- (d) $E_{\alpha, \alpha}(z) \geq 0$ for $z \in \mathbb{R}$.

- (e) Let $0 < \alpha_0 < \alpha_1 < 1$. Then there exists constants $C^-, C^+ > 0$ depending only on α_0, α_1 such that

$$\frac{C^-}{\Gamma(1-\alpha)} \frac{1}{1-z} \leq E_{\alpha, 1}(z) \leq \frac{C^+}{\Gamma(1-\alpha)} \frac{1}{1-z}, \quad \forall z \leq 0.$$

- (f) We have

$$\frac{d}{dz} E_{\alpha, 1}(z) = \frac{1}{\alpha} E_{\alpha, \alpha}(z).$$

- (g) Let a function $g \in L^1(0, T)$ and $\lambda \in \mathbb{C}$. Then the integral equation

$$u(t) = g(t) + \frac{\lambda}{\Gamma(\alpha)} \int_0^t \frac{u(s)}{(t-s)^{1-\alpha}} ds$$

has a unique solution

$$u(t) = g(t) + \lambda \int_0^t E_{\alpha, \alpha}(\lambda(t-s)^\alpha) g(s) ds$$

in the space $L^1(0, T)$.

- (h) Let $g, \varphi \in L^1(0, T)$ and $\lambda \in \mathbb{R}, \lambda \geq 0$. If φ satisfies the integral inequality

$$\varphi(t) \leq g(t) + \frac{\lambda}{\Gamma(\alpha)} \int_0^t \frac{\varphi(s)}{(t-s)^{1-\alpha}} ds$$

then

$$\varphi(t) \leq g(t) + \lambda \int_0^t E_{\alpha,\alpha}(\lambda(t-s)^\alpha) g(s) ds.$$

Moreover, if $g \in L^p(0, T)$, $1 \leq p \leq \infty$ then

$$\|\varphi\|_{L^p(0,T)} \leq (1 + \lambda T E_{\alpha,\alpha}(\lambda T^\alpha)) \|g\|_{L^p(0,T)}.$$

Proof. Proof of (a): The first inequality of part (a) can be verified directly from the definition of the Mittag-Leffler function. In fact, we have

$$|E_{\alpha,\beta}(z)| \leq \sum_{k \leq 2/a_0} \frac{|z|^k}{\Gamma(k\alpha + \beta)} + \sum_{k > 2/a_0} \frac{|z|^k}{\Gamma(k\alpha + \beta)} := E_1 + E_2.$$

Since $\Gamma(x)$ is increasing as $x > 2$, we obtain

$$E_2 \leq \sum_{k > 2/a_0} \frac{M^k}{\Gamma(ka_0 + b_0)}.$$

For $0 \leq k \leq 2/a_0$, $a_0 \leq \alpha \leq a_0 + 2$, $b_0 \leq \beta \leq b_0 + 2$ we have

$$\begin{aligned} \Gamma(k\alpha + \beta) &= \frac{\Gamma(k\alpha + \beta + 2)}{(k\alpha + \beta)(k\alpha + \beta + 1)} \\ &\geq \frac{\Gamma(ka_0 + b_0 + 2)}{(k\alpha + \beta)(k\alpha + \beta + 1)} \\ &= \frac{(ka_0 + b_0)(ka_0 + b_0 + 1)}{(k\alpha + \beta)(k\alpha + \beta + 1)} \Gamma(ka_0 + b_0) \end{aligned}$$

which gives

$$\Gamma(ka_0 + b_0) \leq \frac{(k\alpha + \beta)(k\alpha + \beta + 1)}{(ka_0 + b_0)(ka_0 + b_0 + 1)} \Gamma(k\alpha + \beta) \leq C_E \Gamma(k\alpha + \beta),$$

where

$$C_E = \frac{(2(a_0 + 2)/a_0 + b_0 + 2)(2(a_0 + 2)/a_0 + b_0 + 3)}{b_0(b_0 + 1)}.$$

For $0 \leq k \leq 2/a_0$, $a_0 \leq \alpha \leq a_0 + 2$, $\beta > b_0 + 2$ we have

$$\Gamma(ka_0 + b_0) \leq C_E \Gamma(ka_0 + b_0 + 2) \leq C_E \Gamma(k\alpha + \beta).$$

So we obtain

$$\Gamma(ka_0 + b_0) \leq C_E \Gamma(k\alpha + \beta) \quad \forall 0 \leq k \leq 2/a_0, a_0 \leq \alpha \leq a_0 + 2, \beta \geq b_0.$$

Finally, for $1 \leq k \leq 2/a_0$, $a_0 + 2 < \alpha$, $\beta \geq b_0$ we have $k\alpha + \beta \geq k(a_0 + 2) + b_0 \geq 2$. Hence

$$\Gamma(ka_0 + b_0) \leq C_E \Gamma(k(a_0 + 2) + b_0) \leq C_E \Gamma(k\alpha + \beta).$$

Combining all cases gives

$$E_1 \leq \sum_{k \leq 2/a_0} C_E \frac{M^k}{\Gamma(ka_0 + b_0)}.$$

From the estimate for E_1, E_2 and the inequality $C_E \geq 1$, we obtain

$$|E_{\alpha,\beta}(z)| \leq E_1 + E_2 \leq C_E E_{a_0,b_0}(M), \quad \forall z \in \mathbb{C}, |z| \leq M.$$

To prove Part (b) and the second inequality of Part (a) of the lemma, we make use of some preliminary results. For $0 < \alpha_0 < \alpha_1 < 2$, $\alpha_1 < 2\alpha_0$ we have

$$0 < \frac{\pi\alpha_1}{2} < \min\{\pi, \pi\alpha_0\}.$$

Hence, we can choose φ satisfying

$$0 < \frac{\pi\alpha_1}{2} < \varphi < \min\{\pi, \pi\alpha_0\} \leq \pi.$$

Put $\varphi_0 = \frac{\pi\alpha_1}{2}$ and choose $0 < \rho_0 < \rho_1$. For $\varphi \in (\varphi_0, \pi]$, $\rho \in [\rho_0, \rho_1]$, define the curve

$$\gamma_{\rho,\varphi} = C_{\rho,\varphi}^- \cup C_{\rho,\varphi} \cup C_{\rho,\varphi}^+ \quad (2.2)$$

where

$$\begin{aligned} C_{\rho,\varphi}^+ &= \{re^{i\varphi} : r \geq \rho\}, \\ C_{\rho,\varphi}^- &= \{re^{-i\varphi} : r \geq \rho\}, \\ C_{\rho,\varphi} &= \{\rho e^{i\theta} : -\varphi < \theta < \varphi\}. \end{aligned}$$

Put

$$\begin{aligned} I_{1,\rho}(\alpha, \beta, z) &= \frac{1}{2\alpha\pi i} \int_{C_{\rho,\varphi}^+} \frac{\zeta^{(1-\beta)\alpha} e^{\zeta^{1/\alpha}}}{\zeta - z} d\zeta, \\ I_{2,\rho}(\alpha, \beta, z) &= -\frac{1}{2\alpha\pi i} \int_{C_{\rho,\varphi}^-} \frac{\zeta^{(1-\beta)\alpha} e^{\zeta^{1/\alpha}}}{\zeta - z} d\zeta, \\ I_{3,\rho}(\alpha, \beta, z) &= \frac{1}{2\alpha\pi i} \int_{C_{\rho,\varphi}} \frac{\zeta^{(1-\beta)\alpha} e^{\zeta^{1/\alpha}}}{\zeta - z} d\zeta. \end{aligned}$$

For every $z \in \mathbb{R}$ and $|z - \rho| > \rho_0$ we can find a $C = C(\varphi_0, \rho_0, \rho_1)$ such that

$$\frac{1}{|\zeta - z|} \leq \frac{C}{1 + |z|} \quad \text{for every } \zeta \in \gamma_{\rho,\varphi}. \quad (2.3)$$

Using Lemma 2.2, we can verify directly by the Lebesgue dominated convergence theorem that the functions $I_{j,\rho}$, $j = 1, 2, 3$, are differentiable with respect to the parameters α, β, z . Moreover, for $j = 1, 2, 3$ we claim that

$$\begin{aligned} |I_{j,\rho}(\alpha, \beta, z)| + \left| \frac{\partial I_{j,\rho}}{\partial \alpha}(\alpha, \beta, z) \right| + \left| \frac{\partial I_{j,\rho}}{\partial \beta}(\alpha, \beta, z) \right| &\leq \frac{C}{1 + |z|}, \\ \left| \frac{\partial I_{j,\rho}}{\partial z}(\alpha, \beta, z) \right| &\leq \frac{C}{(1 + |z|)^2}, \end{aligned}$$

where $z \in \mathbb{R}$, $z \leq z_0$, $C = C(\varphi_0, \rho_0, \rho_1, \alpha_0, \alpha_1, \beta_0, \beta_1, z_0)$. Here, to figure out the idea of proving, we give an outline of the proof for the claim. We note that

$$\left| \exp(r^{1/\alpha} e^{\pm i\varphi/\alpha}) \right| = \exp\left(r^{1/\alpha} \cos\left(\frac{\varphi}{\alpha}\right)\right).$$

We have

$$\begin{aligned}
I_{1,\rho}(\alpha, \beta, z) &= \frac{e^{i\varphi}}{2\alpha\pi i} \int_{\rho}^{\infty} \frac{r^{(1-\beta)\alpha} e^{i\varphi(1-\beta)\alpha} \exp(r^{1/\alpha} e^{i\varphi/\alpha})}{r e^{i\varphi} - z} dr, \\
I_{2,\rho}(\alpha, \beta, z) &= -\frac{e^{-i\varphi}}{2\alpha\pi i} \int_{\rho}^{\infty} \frac{r^{(1-\beta)\alpha} e^{-i\varphi(1-\beta)\alpha} \exp(r^{1/\alpha} e^{-i\varphi/\alpha})}{r e^{i\varphi} - z} dr, \\
I_{3,\rho}(\alpha, \beta, z) &= \frac{1}{2\alpha\pi} \int_{-\varphi}^{\varphi} \frac{\rho^{(1-\beta)\alpha} e^{i(1-\beta)\alpha\theta} \exp(\rho e^{i\theta/\alpha})}{\rho e^{i\theta} - z} \rho e^{i\theta} d\theta.
\end{aligned}$$

Now, we have

$$|I_1(\alpha, \beta, z)| \leq \frac{1}{2\alpha_0\pi} \int_{\rho}^{\infty} \frac{r^{(1-\beta)\alpha} g_0(r)}{|r e^{i\varphi} - z|} dr.$$

Using Lemma 2.2 and (2.3), we obtain

$$|I_{1,\rho}(\alpha, \beta, z)| \leq \frac{C}{2\alpha_0\pi} \int_{\rho}^{\infty} \frac{r^{(1+|\beta_0|)\alpha_1} g_0(r)}{1 + |z|} dr.$$

It follows that

$$|I_{1,\rho}(\alpha, \beta, z)| \leq \frac{C}{1 + |z|} \quad \text{for } z \in \mathbb{R}.$$

Similarly

$$|I_{2,\rho}(\alpha, \beta, z)| \leq \frac{C}{1 + |z|} \quad \text{for } z \in \mathbb{R}.$$

Now, we estimate $I_{3,\rho}(\alpha, \beta, z)$. We have

$$|I_{3,\rho}(\alpha, \beta, z)| \leq \frac{\rho}{2\alpha\pi} \int_{-\varphi}^{\varphi} \frac{\rho^{(1-\beta)\alpha} \exp(\rho^{1/\alpha} \cos(\frac{\theta}{\alpha}))}{|\rho e^{i\theta} - z|} d\theta \leq \frac{1}{2\alpha\pi} \int_{-\varphi}^{\varphi} \frac{e}{|\rho e^{i\theta} - z|} d\theta.$$

From (2.3), we have therefore

$$|I_{3,\rho}(\alpha, \beta, z)| \leq \frac{C}{1 + |z|} \quad \text{for every } z \in \mathbb{R}, |z - \rho| \geq \epsilon_0 > 0.$$

Now we consider the derivatives of $I_{j,\rho}$, $j = 1, 2, 3$. We first consider $I_{1,\rho}(\alpha, \beta, z)$. Put the integrand of $I_1(\alpha, \beta)$ by

$$F(\alpha, \beta, r, z) = \frac{r^{(1-\beta)\alpha} e^{i\varphi(1-\beta)\alpha} \exp(r^{1/\alpha} e^{i\varphi/\alpha})}{r e^{i\varphi} - z}.$$

We have

$$\begin{aligned}
(re^{i\varphi} - z) \frac{\partial F}{\partial \alpha}(\alpha, \beta, r, z) &= r^{(1-\beta)\alpha} (1 - \beta) e^{i\varphi(1-\beta)\alpha} \exp(r^{1/\alpha} e^{i\varphi/\alpha}) \ln r \\
&\quad + r^{(1-\beta)\alpha} i\varphi (1 - \beta) e^{i\varphi(1-\beta)\alpha} \exp(r^{1/\alpha} e^{i\varphi/\alpha}) \\
&\quad + r^{(1-\beta)\alpha} e^{i\varphi(1-\beta)\alpha} \left(-\frac{1}{\alpha^2} \right) (r^{1/\alpha} e^{i\varphi/\alpha} \ln r + r^{1/\alpha} i\varphi e^{i\varphi/\alpha}) \exp(r^{1/\alpha} e^{i\varphi/\alpha}).
\end{aligned}$$

Using Lemma 2.2, we get after some rearrangements

$$(1 + |z|) \left| \frac{\partial F}{\partial \alpha}(\alpha, \beta, r, z) \right| \leq C(1 + \ln r) r^{1+|\beta_0|\alpha_1} g_0(r) \quad \text{for } r \geq 1.$$

As mentioned in Lemma 2.2, the function in the left hand side of the latter inequality is Lebesgue integrable on $[1, \infty)$. Hence, the Lebesgue dominated convergence theorem gives

$$\frac{\partial I_{1,\rho}}{\partial \alpha}(\alpha, \beta, z) = \frac{e^{i\varphi}}{2\alpha\pi i} \int_1^\infty \frac{\partial F}{\partial \alpha}(\alpha, \beta, r, z) dr - \frac{e^{i\varphi}}{2\alpha^2\pi i} \int_1^\infty F(\alpha, \beta, r, z) dr$$

and

$$\left| \frac{\partial I_{1,\rho}}{\partial \alpha}(\alpha, \beta, z) \right| \leq \frac{C}{1+|z|}.$$

Similarly, we can get

$$\left| \frac{\partial I_{1,\rho}}{\partial \beta}(\alpha, \beta, z) \right| \leq \frac{C}{1+|z|}, \quad \left| \frac{\partial I_{1,\rho}}{\partial z}(\alpha, \beta, z) \right| \leq \frac{C}{(1+|z|)^2}.$$

Using the same argument, we can prove analogous inequality for $I_{2,\rho}(\alpha, \beta, z), I_{3,\rho}(\alpha, \beta, z)$. Combining the inequalities thus obtained, we get the desired results.

Choosing $\rho > z_0$, we have in view of Theorem 1.1 in [27, Chap. 1, page 30]

$$E_{\alpha,\beta}(z) = \frac{1}{2\alpha\pi i} \int_{\gamma_{\rho,\varphi}} \frac{\zeta^{(1-\beta)\alpha} e^{\zeta^{1/\alpha}}}{\zeta - z} d\zeta \quad \text{for all } z < z_0.$$

It follows that

$$E_{\alpha,\beta}(z) = I_{1,\rho}(\alpha, \beta, z) + I_{2,\rho}(\alpha, \beta, z) + I_{3,\rho}(\alpha, \beta, z).$$

Combining the inequalities for $I_{1,\rho}(\alpha, \beta, z), I_{2,\rho}(\alpha, \beta, z), I_{3,\rho}(\alpha, \beta, z)$ gives the stated results. To prove the Lipschitz property, we use the mean value theorem and the proved inequalities to obtain

$$|E_{\alpha,\beta}(z_1) - E_{\alpha,\beta}(z_2)| \leq \sup_{z \leq z_0} \left| \frac{\partial E_{\alpha,\beta}}{\partial z}(z) \right| |z_1 - z_2| \leq C|z_1 - z_2|.$$

Proof of (b): For $z \geq z_1$, we choose a number $\rho_1 \in (0, z_1)$. We have in view of Theorem 1.1 [27, Chap. 1, page 30]

$$E_{\alpha,\beta}(z) = \frac{1}{\alpha} z^{(1-\beta)\alpha} e^{z^{1/\alpha}} + \frac{1}{2\alpha\pi i} \int_{\gamma_{\rho_1,\varphi}} \frac{\zeta^{(1-\beta)\alpha} e^{\zeta^{1/\alpha}}}{\zeta - z} d\zeta \quad \text{for all } z > z_1.$$

Using the part (a), we obtain (b).

Proof of (c): Put $G(z, \alpha, \beta) = E_{\alpha,\beta}(z) \phi_0(\alpha, \beta, z)^{-1}$. From Part (b) we have for $z \geq 1$

$$\begin{aligned} |G(z, \alpha, \beta) - 1| &\leq \frac{C \max\{z^{(1-\beta_0)\alpha}, z^{(1-\beta_1)\alpha}\}}{(1+|z|)e^{z^{1/\alpha}}} \\ &\leq \frac{C(z^{(1-\beta_0)\alpha} + z^{(1-\beta_1)\alpha})}{(1+|z|)e^{z^{1/\alpha}}} \\ &\leq \frac{C(z^{(1-\beta_0)\alpha_0} + z^{(1-\beta_0)\alpha_1} + z^{(1-\beta_1)\alpha_0} + z^{(1-\beta_1)\alpha_1})}{(1+|z|)e^{z^{1/\alpha_1}}} := \psi(z). \end{aligned}$$

Since $\lim_{z \rightarrow +\infty} \psi(z) = 0$, we can find an $M > z_1 > 0$ independent of z, α, β such that $0 \leq \psi(z) \leq \frac{1}{2}$ for $z \geq M$. It follows that

$$\frac{1}{2} \leq G(z, \alpha, \beta) \leq \frac{3}{2} \quad \text{for } z \geq M.$$

Now, put $D = [z_1, M] \times [\alpha_0, \alpha_1] \times [\beta_0, \beta_1]$, $c^- = \inf_D G(z, \alpha, \beta)$, $c^+ = \sup_D G(z, \alpha, \beta)$. Using compactness argument, we obtain

$$c^- = \min_D G(z, \alpha, \beta) > 0, \quad c^+ = \max_D G(z, \alpha, \beta) > 0$$

and

$$c^- \leq G(z, \alpha, \beta) \leq c^+ \quad \text{for } (z, \alpha, \beta) \in D.$$

Putting $C^- = \min\{c^-, \frac{1}{2}\}$, $C^+ = \max\{c^+, \frac{3}{2}\}$, we have

$$C^- \leq G(z, \alpha, \beta) \leq C^+ \quad \text{for } z \geq z_1.$$

Hence the desired result follows. The case $\beta = 1$ is similar (in fact, easier).

Proof of (d): From the definition we have $E_{\alpha, \alpha}(z) \geq 0$ for $z \geq 0$. For $z < 0$, the proof can be found in [25] which uses the fact that the Mittag-Leffler function is completely monotonic.

Proof of (e): The proof can be found in Simon [32] for the inequality.

Proof of (f): We have

$$E_{\alpha, 1}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha + 1)} \quad z \in \mathbb{C}.$$

Hence

$$\frac{d}{dz} E_{\alpha, 1}(z) = \sum_{k=1}^{\infty} \frac{kz^{k-1}}{k\alpha\Gamma(k\alpha)} = \frac{1}{\alpha} \sum_{k=1}^{\infty} \frac{z^{k-1}}{\Gamma(k\alpha)} = \frac{1}{\alpha} E_{\alpha, \alpha}(z).$$

Proof of (g): See [15, page 63].

Proof of (h): Putting

$$\psi(t) = \varphi(t) - \frac{\lambda}{\Gamma(\alpha)} \int_0^t \frac{\varphi(s)}{(t-s)^{1-\alpha}} ds,$$

we obtain $\psi(t) \leq g(t)$. We deduce in view of the part (g)

$$\begin{aligned} \varphi(t) &= \psi(t) + \lambda \int_0^t E_{\alpha, \alpha}(\lambda(t-s)^\alpha) \psi(s) ds \\ &\leq g(t) + \lambda \int_0^t E_{\alpha, \alpha}(\lambda(t-s)^\alpha) g(s) ds. \end{aligned}$$

Now, we prove that last inequality of the lemma. The case $p = 1$ and $p = \infty$ can be proved easily. Hence we omit it. We consider the case $1 < p < \infty$, $g \in L^p(0, T)$. Putting q such that $\frac{1}{p} + \frac{1}{q} = 1$, using Hölder's inequality we can estimate directly

$$\begin{aligned} \varphi(t) &\leq g(t) + \lambda \int_0^t E_{\alpha, \alpha}(\lambda(t-s)^\alpha) g(s) ds \\ &\leq g(t) + \lambda E_{\alpha, \alpha}(\lambda T^\alpha) \int_0^t g(s) ds \\ &\leq g(t) + \lambda E_{\alpha, \alpha}(\lambda T^\alpha) t^{1/q} \left(\int_0^t |g(s)|^p ds \right)^{1/p} \\ &\leq g(t) + \lambda E_{\alpha, \alpha}(\lambda T^\alpha) T^{1/q} \|g\|_{L^p(0, T)}. \end{aligned}$$

So we have

$$\begin{aligned} \|\varphi\|_{L^p(0, T)} &\leq \|g\|_{L^p(0, T)} + \lambda E_{\alpha, \alpha}(\lambda T^\alpha) T^{1/q} \|g\|_{L^p(0, T)} T^{1/p} \\ &\leq (1 + \lambda T E_{\alpha, \alpha}(\lambda T^\alpha)) \|g\|_{L^p(0, T)}. \end{aligned}$$

□

We note that the final inequality of the lemma will be used to estimate the solution of fractional problems in many cases. This inequality is an extension of the Gronwall inequality. Some other results of interest for the Mittag-Leffler functions can be found in [15] or [22].

3 Continuity of the solutions of some fractional differential equations

In this section, we will investigate the continuity of solutions of a class of general fractional differential equations with respect to the fractional parameter $\alpha \in (0, 1)$. First, we will transform these equations into general Abel equations and then study their properties. The investigation on fractional differential equations with sequential derivatives can be applied directly to other equations with the Riemann-Liouville, or the Caputo fractional derivatives. Hence, we start with the general equation. Let $\sigma_0 = 0 < \sigma_1 < \dots < \sigma_k$ satisfy $0 < \sigma_j - \sigma_{j-1} \leq 1$, $j = 1, \dots, k$. The fractional differential equation with sequential derivatives reads as

$$\mathcal{D}_t^{\sigma_k} y(t) + \sum_{j=1}^k p_j(t) \mathcal{D}_t^{\sigma_{k-j}} y(t) + p_k(t) y(t) = f(t), \quad 0 < t \leq T, \quad (3.1)$$

subject to the conditions

$$\mathcal{D}_t^{\sigma_j-1} y(t) \Big|_{t=0} = b_j, \quad j = 1, \dots, k.$$

In view of [27, page 122], the solution of the equation $\mathcal{D}_t^{\sigma_k} y(t) = \psi(t)$ is given by

$$y(t) = \sum_{j=1}^k b_j \frac{t^{\sigma_j-1}}{\Gamma(\sigma_j)} + \frac{1}{\Gamma(\sigma_k)} \int_0^t (t-s)^{\sigma_k-1} \psi(s) ds.$$

Using this equality we can rewrite the equation (3.1) as

$$\psi(t) + \int_0^t \frac{K(t, s, \beta, \psi(s))}{(t-s)^{1-\sigma_k}} ds = g(t) \quad (3.2)$$

where $\eta_j := \sigma_j - \sigma_{j-1}$, $\beta := (\eta_1, \dots, \eta_k)$ and

$$\begin{aligned} K(t, s, \beta, \psi) &= \left(p_k(t) \frac{(t-s)^{\sigma_k-\eta_k}}{\Gamma(\sigma_k)} + \sum_{j=1}^{k-1} p_{k-j}(t) \frac{(t-s)^{\sigma_k-\sigma_j-\eta_k}}{\Gamma(\sigma_k)} \right) \psi, \\ g(t) &= f(t) - p_k(t) \sum_{j=1}^k \frac{b_j t^{\sigma_j-1}}{\Gamma(\sigma_j)} - \sum_{j=1}^{k-1} p_{k-j}(t) \sum_{\ell=j+1}^k \frac{b_\ell t^{\sigma_\ell-\sigma_j-1}}{\Gamma(\sigma_\ell - \sigma_j)}. \end{aligned}$$

Put $\nu = \min\{\eta_1, \dots, \eta_k\}$, under mild conditions on the functions f , and p_k 's the function $g^*(t) = t^{1-\nu} g(t)$ is a continuous function on $[0, T]$. By this fact, we consider our problem in the following space

$$C_\gamma(T, \mathbb{X}) = \left\{ v \in C((0, T], \mathbb{X}) : \sup_{0 < t \leq T} t^\gamma \|v(t)\| < \infty \right\}$$

where $0 < \gamma < 1$, $(\mathbb{X}, \|\cdot\|)$ is a Banach space. The space $C_\gamma(T, \mathbb{X})$ is a Banach space with the norm

$$\|v\|_{C_\gamma} = \sup_{0 < t \leq T} t^\gamma \|v(t)\|.$$

From the definition of the norm, we deduce an inequality which will be used often in the the rest of our paper

$$\|v(t)\| \leq t^{-\gamma} \|v\|_{C_\gamma(T)}.$$

For convenience, we denote $C([0, T]; \mathbb{X})$ by $C_0(T; \mathbb{X})$.

3.1 Some properties of solutions of the generalized Abel equations of the second kind

We will establish existence and continuity of solutions of the genralized Abel equations of second kind. The main results are given in Theorems 3.1 and 3.2.

For $k \in \mathbb{N}$, we denote by P a compact subset in \mathbb{R}^k . Letting $T > 0, 0 < \alpha_0 < \alpha_1$, we put

$$\Delta_T = \left\{ (t, s, \alpha, z) : 0 \leq s \leq t \leq T, \alpha_0 \leq \alpha \leq \alpha_1, z \in P \right\}.$$

Assume that

$$K : \Delta_T \times \mathbb{X} \rightarrow \mathbb{X}, \quad g : (0, T] \times [\alpha_0, \alpha_1] \times P \rightarrow \mathbb{X}.$$

Suggested by the integral form of the general fractional differential equations, we consider the nonlinear Abel integral equation of second kind of finding $u_{\alpha, z} : (0, T] \rightarrow \mathbb{X}$ that satisfy the following

$$u_{\alpha, z}(t) = g(t, \alpha, z) + \int_0^t \frac{K(t, s, \alpha, z, u_{\alpha, z}(s))}{(t-s)^{1-\alpha}} ds. \quad (3.3)$$

For every $\alpha \in [\alpha_0, \alpha_1], v \in C_\gamma(T, \mathbb{X})$, we put

$$A_{\alpha, z}v(t) = \int_0^t \frac{K(t, s, \alpha, z, v(s))}{(t-s)^{1-\alpha}} ds.$$

To emphasize the dependence on the parameter (α, z) , we also denote $g(t, \alpha, z)$ by $g_{\alpha, z}(t)$. With these notations, we can write the equation (3.3) as

$$u_{\alpha, z}(t) = g_{\alpha, z}(t) + A_{\alpha, z}u_{\alpha, z}(t). \quad (3.4)$$

For $p, q > 0$, recalling the definition of Beta function $B(p, q) = \int_0^1 (1-\theta)^{p-1} \theta^{q-1} d\theta$, we have

$$\int_0^t (t-s)^{p-1} s^{q-1} ds = t^{p+q-1} B(p, q) = t^{p+q-1} \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}.$$

In the next lemma we establish some estimates of singular integrals.

Lemma 3.1. (a) *For $h \in [0, 1], T > 0, 0 < \nu_0 \leq \nu_1, \eta, \nu \in [\nu_0, \nu_1], p \geq 1$, there exists a constant $C = C(\nu_0, \nu_1, T)$ such that*

$$\int_0^t s^{\eta-1} (t-s)^{\nu-1} \left(|s^h - 1|^p + |(t-s)^h - 1|^p \right) ds \leq Ch t^{\eta+\nu-1} (1 + |\ln t|^p) \quad \text{for every } 0 \leq t \leq T.$$

(b) *Let $t \in (0, T]$. For $v \in C_\gamma(T; \mathbb{X})$, we have*

$$\|J^\alpha v(t)\| \leq J^\alpha \|v(t)\| \leq t^{\alpha-\gamma} \frac{\|v\|_{C_\gamma(T; \mathbb{X})}}{\Gamma(\alpha)} B(\alpha, 1-\gamma)$$

and for $v \in L^p(0, T; X)$, $1 \leq p \leq \infty$,

$$\|J^\alpha v(t)\|_{L^p(0, T; \mathbb{X})} \leq \|J^\alpha \|v(t)\| \|_{L^p(0, T)} \leq \frac{T^\alpha}{\alpha \Gamma(\alpha)} \|v\|_{L^p(0, T; \mathbb{X})}.$$

Let $\mu_0, h \in (0, 1]$. If $v \in C([0, T]; \mathbb{X})$ is Hölder, i.e., there exist $\kappa > 0, \mu \in (\mu_0, 1)$ such that $\|v(t) - v(s)\| \leq \kappa |t - s|^\mu$ for every $t, s \in [0, T]$, then there exists a constant $C(\mu_0)$ independent of v, μ such that

$$\|J^h v(s) - v(s)\| \leq C(\mu_0)(\|v\|_{C([0, T]; \mathbb{X})} + [v]_\mu)(h + |s^h - 1|)$$

where $[v]_\mu = \sup_{0 \leq t \neq s \leq T} \frac{\|v(t) - v(s)\|}{|t - s|^\mu}$. We also have

$$\begin{aligned} \lim_{h \rightarrow 0^+} \|J^h w - w\|_{L^p(0, T; \mathbb{X})} &= 0, \quad \forall p \in [1, \infty), w \in L^p(0, T; \mathbb{X}), \\ \|J^{\alpha'} w - J^\alpha w\|_{L^p(0, T; \mathbb{X})} &\leq C(\alpha_0) \|w\|_{L^p(0, T; \mathbb{X})} |\alpha' - \alpha|, \quad \forall \alpha_0 \in (0, 1], \alpha', \alpha \in [\alpha_0, 1]. \end{aligned}$$

(c) Let $K \in C(\Delta_T \times \mathbb{X}; \mathbb{X})$, $K = K(t, s, \alpha, z, w)$. We assume that K is Lipschitz with respect to the variable $w \in \mathbb{X}$, i.e., there exists a $\kappa > 0$ such that

$$\|K(t, s, \alpha, z, w_1) - K(t, s, \alpha, z, w_2)\| \leq \kappa \|w_1 - w_2\| \quad \text{for every } w_1, w_2 \in \mathbb{X}. \quad (3.5)$$

Put $M_0 = \sup_{(t, s, \alpha, z) \in \Delta_T} \|K(t, s, \alpha, z, 0)\|$. For $0 < t \leq T$, $v, v_1, v_2 \in C_\gamma(T, \mathbb{X})$ we have $A_\alpha v \in C_\gamma(T, \mathbb{X})$. Moreover, we have

$$\begin{aligned} \|A_{\alpha, z} v(t)\| &\leq \frac{M_0}{\alpha_0} t^\alpha + \kappa t^{\alpha - \gamma} B(\alpha, 1 - \gamma) \|v\|_{C_\gamma(T, \mathbb{X})}, \\ \|A_{\alpha, z} v_1(t) - A_{\alpha, z} v_2(t)\| &\leq \kappa t^{\alpha - \gamma} B(\alpha, 1 - \gamma) \|v_1 - v_2\|_{C_\gamma(T, \mathbb{X})} \end{aligned}$$

and

$$\|A_{\alpha, z} v_1 - A_{\alpha, z} v_2\|_{C_\gamma(T, \mathbb{X})} \leq \kappa T^\alpha B(\alpha, 1 - \gamma) \|v_1 - v_2\|_{C_\gamma(T, \mathbb{X})}.$$

(d) Now, let $v \in L^p(0, T; \mathbb{X})$. For $1 \leq p < \infty$, we have

$$\|A_{\alpha, z} v\|_{L^p(0, T; \mathbb{X})} \leq \frac{M_0}{\alpha(p\alpha + 1)^{1/p}} T^{\alpha + 1/p} + \frac{\kappa T^\alpha}{\alpha} \|v\|_{L^p(0, T; \mathbb{X})}.$$

For $p = \infty$, we have

$$\|A_{\alpha, z} v\|_{L^\infty(0, T; \mathbb{X})} \leq \frac{M_0}{\alpha} T^\alpha + \frac{\kappa T^\alpha}{\alpha} \|v\|_{L^p(0, T; \mathbb{X})}.$$

For $v_1, v_2 \in L^p(0, T; \mathbb{X})$, $1 \leq p \leq \infty$, we also have

$$\|A_{\alpha, z} v_1 - A_{\alpha, z} v_2\|_{L^p(0, T; \mathbb{X})} \leq \frac{\kappa T^\alpha}{\alpha} \|v_1 - v_2\|_{L^p(0, T; \mathbb{X})}.$$

(e) Let $u_1, u_2, g_1, g_2 \in L^p(0, T; \mathbb{X})$ satisfy the equations $u_i = g_i + A_{\alpha, z} u_i$, $i = 1, 2$. Then we have

$$\|u_2 - u_1\|_{L^p(0, T; \mathbb{X})} \leq (1 + \kappa \Gamma(\alpha) T E_{\alpha, \alpha}(\kappa \Gamma(\alpha) T^\alpha)) \|g_2 - g_1\|_{L^p(0, T; \mathbb{X})}.$$

Proof. **Proof of (a):** The proof follows as

$$\begin{aligned}
& \int_0^t s^{\eta-1} (t-s)^{\nu-1} (|s^h - 1|^p + |(t-s)^h - 1|^p) ds \\
&= \int_0^t s^{\eta-1} (t-s)^{\nu-1} \left(\left| \int_0^h s^\mu \ln s d\mu \right|^p + \left| \int_0^h (t-s)^\mu \ln(t-s) d\mu \right|^p \right) ds \\
&\leq \int_0^t s^{\eta-1} (t-s)^{\nu-1} \left(\int_0^h s^\mu |\ln s| d\mu + \int_0^h (t-s)^\mu |\ln(t-s)| d\mu \right)^p ds \\
&\leq C_0 h \int_0^t s^{\eta-1} (t-s)^{\nu-1} (|\ln s|^p + |\ln(t-s)|^p) ds \\
&\leq C_0 t^{\eta+\nu-1} h \int_0^1 \theta^{\eta-1} (1-\theta)^{\nu-1} (|\ln \theta|^p + |\ln(1-\theta)|^p + 2|\ln t|^p) d\theta \\
&\leq C h t^{\eta+\nu-1} (1 + |\ln t|^p).
\end{aligned}$$

Proof of (b): We verify the inequalities in Part (b). We have

$$\begin{aligned}
\|J^\alpha v(t)\| &\leq J^\alpha \|v(t)\| \\
&= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} s^{-\gamma} s^\gamma \|v(s)\| ds \\
&\leq \frac{\|v\|_{C_\gamma(T;\mathbb{X})}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} s^{(1-\gamma)-1} ds \\
&= \frac{\|v\|_{C_\gamma(T;\mathbb{X})}}{\Gamma(\alpha)} t^{\alpha-\gamma} B(\alpha, 1-\gamma).
\end{aligned}$$

Now, if $v \in L^p(0, T; \mathbb{X})$, since we can prove the cases $p = 1$ and $p = \infty$ easily, we only consider the case $1 < p < \infty$. Let q be such that $\frac{1}{p} + \frac{1}{q} = 1$, we get

$$\Gamma(\alpha) \| (J^\alpha \|v(t)\|) \|_{L^p(0, T)} \leq \int_0^t \frac{ds}{(t-s)^{1-\alpha}} \|v\|_{L^p(0, T; \mathbb{X})} \leq \frac{T^\alpha}{\alpha} \|v\|_{L^p(0, T; \mathbb{X})}.$$

Now, we consider the limit of $J^h v$ as $h \rightarrow 0^+$. Put $I(s) = \|J^h v(s) - v(s)\|$ and $e(\tau) = \frac{1}{\tau} \int_{s-\tau}^s v(\theta) d\theta - v(s)$. Using integration by parts we can write

$$\begin{aligned}
I(s) &= \frac{1}{\Gamma(h)} \int_0^s (s-\tau)^{h-1} v(\tau) d\tau - v(s) \\
&= \frac{1}{\Gamma(h)} s^{h-1} \int_0^s v(\tau) d\tau + \frac{1-h}{\Gamma(h)} \int_0^s \tau^{h-2} \int_{s-\tau}^s v(\theta) d\theta d\tau - v(s) \\
&= \frac{1}{\Gamma(h)} s^{h-1} \int_0^s v(\tau) d\tau + \frac{1-h}{\Gamma(h)} \int_0^s \tau^{h-1} e(\tau) d\tau \\
&\quad + \left(\frac{1-h}{\Gamma(1+h)} s^h - 1 \right) v(s).
\end{aligned}$$

Next we estimate $e(\tau)$. Letting $\theta = s - \eta\tau$ gives

$$\begin{aligned}
\|e(\tau)\| &= \left\| \int_0^1 (v(s - \eta\tau) - v(s)) d\eta \right\| \\
&\leq [v]_\mu \int_0^1 |s - (s - \tau\eta)|^\mu d\eta \leq [v]_\mu \tau^\mu.
\end{aligned}$$

Therefore

$$\|I(s)\| \leq Ch\|v\|_{C([0,T];\mathbb{X})} + C\frac{[v]_\mu(1-h)}{\Gamma(h)} \int_0^s \tau^{h-1+\mu} d\tau + \left(\frac{1-h}{\Gamma(1+h)} s^h - 1 \right) \|v(s)\|.$$

By direct computation, we get

$$\|I(s)\| \leq C \left(h + |1 - s^h| \right) (\|v\|_{C([0,T];\mathbb{X})} + [v]_\mu).$$

Now, we prove the final equality of Part (b). Let $\epsilon > 0$, since $C^1([0, T]; \mathbb{X})$ is dense in $L^p(0, T; \mathbb{X})$, we can find a function $v \in C^1([0, T]; \mathbb{X})$ such that $\|w - v\|_{L^p(0, T; \mathbb{X})} \leq \epsilon$. From the estimate of $I(s)$ we obtain

$$\begin{aligned} \|J^h w - w\|_{L^p(0, T; \mathbb{X})} &\leq \|J^h(w - v)\|_{L^p(0, T; \mathbb{X})} + \|J^h v - v\|_{L^p(0, T; \mathbb{X})} + \\ &\quad \|v - w\|_{L^p(0, T; \mathbb{X})} \\ &\leq C\epsilon + C\|v\|_{C^1([0, T]; \mathbb{X})} \left(h + \left(\int_0^T |1 - s^h|^p ds \right)^{1/p} \right). \end{aligned}$$

Using Part (a), we obtain the claimed limit. Now we prove the last inequality of Part (b). We have in view of Part (a)

$$\begin{aligned} \|J^{\alpha'} w - J^\alpha w\|_{L^p(0, T; \mathbb{X})} &= \left\| \int_0^t \left((t-s)^{\alpha'-1} - (t-s)^{\alpha-1} \right) w(s) ds \right\|_{L^p(0, T; \mathbb{X})} \\ &\leq \sup_{0 < t \leq T} \int_0^t (t-s)^{\alpha-1} \left| (t-s)^{\alpha'-\alpha} - 1 \right| ds \|w\|_{L^p(0, T; \mathbb{X})} \\ &\leq C|\alpha' - \alpha| \sup_{0 < t \leq T} t^\alpha (1 + |\ln t|) \|w\|_{L^p(0, T; \mathbb{X})}. \end{aligned}$$

Since $\alpha \in (\alpha_0, 1]$ we get $\sup_{0 < t \leq T} t^\alpha (1 + |\ln t|) \leq C(\alpha_0) < \infty$. Hence

$$\|J^{\alpha'} w - J^\alpha w\|_{L^p(0, T; \mathbb{X})} \leq C(\alpha_0) |\alpha' - \alpha| \|w\|_{L^p(0, T; \mathbb{X})}.$$

Proof of (c): For $v \in \mathbb{X}$, using equation (3.5) we have

$$\|K(t, s, \alpha, z, v)\| \leq M_0 + \kappa \|v\|. \quad (3.6)$$

Hence, we obtain in view of Part a) and the inequality $\alpha_0 \leq \alpha$ that

$$\begin{aligned} \|A_{\alpha, z} v(t)\| &\leq \Gamma(\alpha) (J^\alpha M_0 + J^\alpha \kappa \|v(t)\|) \\ &\leq \frac{M_0}{\alpha_0} t^\alpha + \kappa t^{\alpha-\gamma} B(\alpha, 1-\gamma) \|v\|_{C_\gamma(T, \mathbb{X})}. \end{aligned}$$

Similarly we obtain the following estimates

$$\begin{aligned} \|A_{\alpha, z} v_1(t) - A_{\alpha, z} v_2(t)\| &\leq \Gamma(\alpha) \kappa J^\alpha \|v_1(t) - v_2(t)\| \leq \kappa t^{\alpha-\gamma} \|v\|_{C_\gamma(T, \mathbb{X})} B(\alpha, 1-\gamma), \\ \|A_{\alpha, \beta} v(t) - A_{\alpha, \beta}(0)\| &\leq \Gamma(\alpha) \kappa J^\alpha \|v(t)\| \leq \kappa t^{\alpha-\gamma} \|v\|_{C_\gamma(T, \mathbb{X})} B(\alpha, 1-\gamma). \end{aligned}$$

Proof of (d): We have

$$\begin{aligned} \|A_{\alpha, z} v\|_{L^p(0, T; \mathbb{X})} &\leq \Gamma(\alpha) \|J^\alpha M_0\|_{L^p(0, T)} + \Gamma(\alpha) \|J^\alpha \kappa \|v(t)\|\|_{L^p(0, T; \mathbb{X})} \\ &\leq \frac{M_0 T^{\alpha+1/p}}{\alpha(p\alpha+1)^{1/p}} + \frac{T^\alpha}{\alpha} \|v\|_{L^p(0, T; \mathbb{X})}. \end{aligned}$$

We also have

$$\|A_{\alpha,z}v_1(t) - A_{\alpha,z}v_2(t)\| \leq \kappa\Gamma(\alpha)J^\alpha\|v_1(t) - v_2(t)\|.$$

Therefore, we obtain in view of Part (a)

$$\|A_{\alpha,z}v_1 - A_{\alpha,z}v_2\|_{L^p(0,T;\mathbb{X})} \leq \frac{\kappa T^\alpha}{\alpha}\|v_1 - v_2\|_{L^p(0,T;\mathbb{X})}.$$

Proof of (e): We have

$$\|u_2(t) - u_1(t)\| \leq \|g_2(t) - g_1(t)\| + \kappa \int_0^t \frac{\|u_2(s) - u_1(s)\|}{(t-s)^{1-\alpha}} ds.$$

Applying Lemma 2.3 (h) with $\lambda = \kappa\Gamma(\alpha)$ gives

$$\|u_2 - u_1\|_{L^p(0,T;\mathbb{X})} \leq (1 + \kappa\Gamma(\alpha)TE_{\alpha,\alpha}(\kappa\Gamma(\alpha)T^\alpha))\|g_2 - g_1\|_{L^p(0,T;\mathbb{X})}.$$

□

The following theorem gives the existence of solutions of the nonlinear integral equation (3.4) in $C_\gamma(T)$ and $L^p(0,T;\mathbb{X})$.

Theorem 3.1. *Let $0 < \alpha_0 < \alpha_1$, $\alpha \in [\alpha_0, \alpha_1]$ and $\gamma \in [0, 1)$. Let $K \in C(\Delta_T \times \mathbb{X}; \mathbb{X})$, $K = K(t, s, \alpha, z, v)$. We assume that K is a Lipschitz function with respect to the variable $v \in \mathbb{X}$, i.e., there exists a $\kappa > 0$ such that*

$$\|K(t, s, \alpha, z, v_1) - K(t, s, \alpha, z, v_2)\| \leq \kappa\|v_1 - v_2\| \quad \text{for every } v_1, v_2 \in \mathbb{X}.$$

(a) *If $g \in C([\alpha_0, \alpha_1] \times P; C_\gamma(T, \mathbb{X}))$ then the nonlinear equation (3.4) has a unique solution $u_{\alpha,z} \in C_\gamma(T)$ such that*

$$\|u_{\alpha,z}\|_{C_\gamma(T,\mathbb{X})} \leq \Gamma(1-\gamma)E_{\alpha,1-\gamma}(\kappa\Gamma(\alpha)T^\alpha)\|g_{\alpha,z}^*\|_{C_\gamma(T,\mathbb{X})}$$

where

$$g_{\alpha,z}^*(t) = g_{\alpha,z}(t) + \int_0^t \frac{K^*(t, s, \alpha, z, 0)}{(t-s)^{1-\alpha}} ds.$$

(a) *If $g_{\alpha,z} \in L^p(0,T;\mathbb{X})$ then the nonlinear equation (3.4) has a unique solution $u_{\alpha,z} \in L^p(0,T;\mathbb{X})$. Moreover, we have the following estimate*

$$\|u_{\alpha,z}\|_{L^p(0,T;\mathbb{X})} \leq \left(1 + \kappa T\Gamma(\alpha)E_{\alpha,\alpha}(\kappa\Gamma(\alpha)T^\alpha)\right)\|g_{\alpha,z}^*\|_{L^p(0,T;\mathbb{X})}.$$

Proof. **Proof of (a):** Putting

$$K^*(t, s, \alpha, z, w) = K(t, s, \alpha, z, w) - K(t, s, \alpha, z, 0),$$

we obtain $K^*(t, s, \alpha, z, 0) \equiv 0$ and, for $w_1, w_2 \in \mathbb{X}$,

$$\|K^*(t, s, \alpha, z, w_1) - K^*(t, s, \alpha, z, w_2)\| \leq \kappa\|w_1 - w_2\|.$$

Defining

$$A_{\alpha,z}^* u = \int_0^t \frac{K^*(t,s,\alpha,z,u(s))}{(t-s)^{1-\alpha}} ds,$$

we can rewrite (3.4) in the operator form

$$(I - A_{\alpha,z}^*) u_{\alpha,z} = g_{\alpha,z}^*.$$

We verify the uniqueness of solution of the problem. Letting $v, w \in C_\gamma(T, \mathbb{X})$ be two solutions of the latter problem, we obtain

$$\begin{aligned} \|v(t) - w(t)\| &\leq \int_0^t \frac{\|K(t,s,\alpha,z,v(s)) - K(t,s,\alpha,z,w(s))\|}{(t-s)^{1-\alpha}} ds \\ &\leq \int_0^t \frac{\kappa \|v(s) - w(s)\|}{(t-s)^{1-\alpha}} ds. \end{aligned}$$

From Lemma 2.3, Part (h), we obtain $\|v(t) - w(t)\| = 0$, or $v = w$.

Now we prove the existence of solution of the problem. Putting $u_0 = 0$, $u_1 = g_{\alpha,z}^*$, $u_{n+1} = g_{\alpha,z}^* + A_{\alpha,z}^* u_n$, we claim that the series u_n converges in $C_\gamma(T, \mathbb{X})$ and its limit is the solution of (3.4). In fact, putting $M_1 = \kappa \Gamma(\alpha)$, we can prove by induction that

$$\|u_n(t) - u_{n-1}(t)\| \leq \frac{\|g_{\alpha,z}^*\|_{C_\gamma(T,\mathbb{X})} \Gamma(1-\gamma) M_1^n t^{n\alpha-\gamma}}{\Gamma(n\alpha+1-\gamma)} \quad \text{for } n = 1, 2, \dots$$

For $n = 1$, we have in view of Lemma 3.1, Part (c),

$$\begin{aligned} \|u_1(t) - u_0(t)\| &\leq \int_0^t \frac{\|K^*(t,s,\alpha,z,g_{\alpha,\beta}^*(s))\|}{(t-s)^{1-\alpha}} ds \\ &\leq \kappa \|g_{\alpha,z}^*\|_{C_\gamma(T,\mathbb{X})} B(\alpha, 1-\gamma) t^{\alpha-\gamma} \\ &\leq \frac{\|g_{\alpha,z}^*\|_{C_\gamma(T,\mathbb{X})} M_1 \Gamma(1-\gamma) t^{\alpha-\gamma}}{\Gamma(n\alpha+1-\gamma)}. \end{aligned}$$

Assume that the inequality holds for $n = k$ ($k \geq 1$), we claim that it holds for $n = k+1$. To this end, we note that

$$\begin{aligned} \|u_{k+1}(t) - u_k(t)\| &\leq \int_0^t \frac{\|K^*(t,s,\alpha,z,u_k(s)) - K^*(t,s,\alpha,z,u_{k-1}(s))\|}{(t-s)^{1-\alpha}} ds \\ &\leq \kappa \int_0^t \frac{\|u_k(s) - u_{k-1}(s)\|}{(t-s)^{1-\alpha}} ds. \end{aligned}$$

Using the induction assumptions, we deduce

$$\begin{aligned} \|u_{k+1}(t) - u_k(t)\| &\leq \frac{\|g_{\alpha,z}^*\|_{C_\gamma(T,\mathbb{X})} \Gamma(1-\gamma) M_1^{k+1}}{\Gamma(\alpha) \Gamma(k\alpha+1-\gamma)} \int_0^t (t-s)^{\alpha-1} s^{k\alpha+(1-\gamma)-1} ds \\ &\leq \frac{\|g_{\alpha,z}^*\|_{C_\gamma(T,\mathbb{X})} \Gamma(1-\gamma) M_1^{k+1}}{\Gamma(\alpha)} \frac{t^{(k+1)\alpha-\gamma} B(\alpha, k\alpha+(1-\gamma))}{\Gamma(k\alpha+1-\gamma)} \\ &\leq \frac{\|g_{\alpha,z}^*\|_{C_\gamma(T,\mathbb{X})} \Gamma(1-\gamma) M_1^{k+1} t^{(k+1)\alpha-\gamma}}{\Gamma((k+1)\alpha+1-\gamma)}. \end{aligned}$$

The induction principle implies that the inequality holds for every $n = 1, 2, \dots$. Hence the last inequality implies

$$\|u_n - u_{n-1}\|_{C_\gamma(T, \mathbb{X})} \leq \frac{\Gamma(1-\gamma)M_1^n T^{n\alpha}}{\Gamma(n\alpha + 1 - \gamma)} \|g_{\alpha,z}^*\|_{C_\gamma(T, \mathbb{X})}.$$

Since

$$\sum_{k=0}^{\infty} \frac{\|g_{\alpha,z}^*\|_{C_\gamma(T, \mathbb{X})} (M_1 T^\alpha)^k}{\Gamma(k\alpha + 1 - \gamma)} = \|g_{\alpha,z}^*\|_{C_\gamma(T, \mathbb{X})} E_{\alpha, 1-\gamma}(M_1 T^\alpha) < \infty,$$

the Weierstrass theorem implies that $u_n = \sum_{k=1}^n (u_k - u_{k-1})$ converges in $C_\gamma(T, \mathbb{X})$ to a function $u_{\alpha,z}$. We can verify directly that $u_{\alpha,z}$ is the solution of the Abel equation and that

$$\|u_{\alpha,z}\|_{C_\gamma} \leq \Gamma(1-\gamma) E_{\alpha, 1-\gamma}(M_1 T^\alpha) \|g_{\alpha,z}^*\|_{C_\gamma(T, \mathbb{X})}.$$

Proof of (b): We shall use the part (a) and an approximation argument. We choose a sequence $g_n \in C_c^1([0, T]; \mathbb{X})$ such that $g_n \rightarrow g_{\alpha,z}^*$ in $L^p(0, T; \mathbb{X})$. From the part (a), there exist a unique solution $u_n \in C_0(T; \mathbb{X})$ such that

$$u_n = g_n + A_{\alpha,z}^* u_n.$$

Moreover, we can find a constant $\epsilon_n > 0$ such that $u_n(t) = 0$ for $t \in [0, \epsilon_n]$. By direct estimating, we obtain

$$\|u_n(t) - u_m(t)\| \leq \|g_n(t) - g_m(t)\| + \kappa \int_0^t \frac{\|u_n(s) - u_m(s)\|}{(t-s)^{1-\alpha}} ds.$$

Using Lemma 2.3, Part (h), we obtain

$$\begin{aligned} \|u_n(t) - u_m(t)\| &\leq \|g_n(t) - g_m(t)\| \\ &\quad + \kappa \Gamma(\alpha) \int_0^t E_{\alpha, \alpha}(\Gamma(\alpha) \kappa (t-s)^\alpha) \|g_n(s) - g_m(s)\| ds \\ &\leq \|g_n(t) - g_m(t)\| \\ &\quad + \kappa \Gamma(\alpha) E_{\alpha, \alpha}(\Gamma(\alpha) \kappa T^\alpha) \int_0^t \|g_n(s) - g_m(s)\| ds. \end{aligned} \tag{3.7}$$

Calculating directly gives

$$\begin{aligned} \|u_n(t) - u_m(t)\| &\leq \|g_n(t) - g_m(t)\| + \\ &\quad \kappa \Gamma(\alpha) E_{\alpha, \alpha}(\Gamma(\alpha) \kappa T^\alpha) T^{1/q} \|g_n(t) - g_m(t)\|_{L^p(0, T)}. \end{aligned}$$

Hence,

$$\|u_n - u_m\|_{L^p(0, T; \mathbb{X})} \leq \left(1 + \kappa T \Gamma(\alpha) E_{\alpha, \alpha}(\Gamma(\alpha) \kappa T^\alpha)\right) \|g_n - g_m\|_{L^p(0, T; \mathbb{X})}.$$

This implies $\{u_n\}$ is Cauchy in $L^p(0, T; \mathbb{X})$. So it has a limit $u_{\alpha,z} \in L^p(0, T; \mathbb{X})$. From Lemma 3.1, Part (d), we have $A_{\alpha,z} u_n \rightarrow A_{\alpha,z} u_{\alpha,z}$ in $L^p(0, T; \mathbb{X})$. Since $u_n = g_{\alpha,z}^* + A_{\alpha,z}^* u_n$, we get

$$u_{\alpha,z} = g_{\alpha,z}^* + A_{\alpha,z}^* u_{\alpha,z}.$$

Using the same estimate as in the proof of existence, we obtain

$$\|u_{\alpha,z}\|_{L^p(0, T; \mathbb{X})} \leq \left(1 + \kappa T \Gamma(\alpha) E_{\alpha, \alpha}(\Gamma(\alpha) \kappa T^\alpha)\right) \|g_{\alpha,z}^*\|_{L^p(0, T; \mathbb{X})}.$$

□

Before stating and proving the main result, we consider a compactness result in $C_\gamma(T, \mathbb{X})$.

Lemma 3.2. *Let $K \in C(\Delta_T \times \mathbb{X}; \mathbb{X})$, $K = K(t, s, \alpha, z, v)$. We assume that*

- (i) *K is Lipschitz with respect to the variable $v \in \mathbb{X}$ as in Theorem 3.1,*
- (ii) *$\lim_{\xi \rightarrow 0^+} \omega_K(\xi, M) = 0$ where*

$$\omega_K(\xi, M) := \sup_B \|K(t_2, s, \alpha, z, v) - K(t_1, s, \alpha, z, v)\|$$

and

$$B = \left\{ (t_1, t_2, s, \alpha, z, w) : |t_2 - t_1| \leq \xi, 0 < s \leq \min\{t_1, t_2\}, \alpha \in [\alpha_0, \alpha_1], z \in P, w \in \mathbb{X}, \|w\| \leq M \right\}.$$

Then the following results hold

- (a) *the set*

$$B(\alpha_0, \alpha_1, P, L) = \left\{ A_{\alpha, z} v : \alpha \in [\alpha_0, \alpha_1], z \in P, v \in C_\gamma(T, \mathbb{X}), \|v\|_{C_\gamma(T, \mathbb{X})} \leq L \right\}$$

is precompact in $C_\gamma(T, \mathbb{X})$.

- (a) *let $a_n, \alpha \in (0, 1), z_n, z \in P, \lim_{n \rightarrow \infty} a_n = \alpha, \lim_{n \rightarrow \infty} z_n = z$. Assume that $w_n, w \in C_\gamma(T, \mathbb{X})$ and $\lim_{n \rightarrow \infty} w_n = w$ in $C_\gamma(T, \mathbb{X})$. Then $A_{a_n, z_n} w_n \rightarrow A_{\alpha, z} w$ in $C_\gamma(T, \mathbb{X})$ as $n \rightarrow \infty$.*

Proof. Proof of (a): To this end, we shall prove that $B(\alpha_0, \alpha_1, P, L)$ is equibounded and equicontinuous on $[\delta, T]$ for every $\delta \in (0, T]$. For $w \in C_\gamma(T, \mathbb{X}), \|w\|_{C_\gamma(T, \mathbb{X})} \leq L$, Lemma 3.1 part (c) implies

$$\begin{aligned} t^\gamma \|A_{\alpha, z} w(t)\| &\leq \frac{M_0}{\alpha_0} t^{\alpha+\gamma} + \kappa L t^\alpha B(\alpha, 1-\gamma) \\ &\leq \max\{T^{\alpha_0}, T^{\alpha_1}\} \left(\frac{M_0}{\alpha_0} T^\gamma + \kappa L \max_{\alpha_0 \leq \alpha \leq \alpha_1} B(\alpha, 1-\gamma) \right). \end{aligned} \quad (3.8)$$

Hence $B(\alpha_0, \alpha_1, P, L)$ is equibounded in $C_\gamma(T, \mathbb{X})$. This implies that $B(\alpha_0, \alpha_1, P, L)$ is equibounded on $C[\delta, T]$. Now, we verify that $B(\alpha_0, \alpha_1, P, L)$ is equicontinuous on $[\delta, T]$. For $\xi > 0$, $\delta \leq t_1 \leq t_2 \leq T$, $|t_2 - t_1| \leq \xi$ we have

$$A_\alpha v(t_2) - A_\alpha v(t_1) = J_1 + J_2 + J_3$$

with

$$\begin{aligned} J_1 &= \int_0^{t_1} ((t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}) K(t_2, s, \alpha, z, v(s)) ds, \\ J_2 &= \int_0^{t_1} (t_1 - s)^{\alpha-1} (K(t_2, s, \alpha, z, v(s)) - K(t_1, s, \alpha, z, v(s))) ds, \\ J_3 &= \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} K(t_2, s, \alpha, z, v(s)) ds. \end{aligned}$$

Using (3.6) we have

$$\begin{aligned} |J_1| &\leq \int_0^{t_1} |(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}| (M_0 + \kappa L s^{-\gamma}) ds \\ &\leq \frac{M_0}{\alpha} (t_2^\alpha - t_1^\alpha) + \kappa L \int_0^{t_1} (t_1 - s)^{\alpha-1} s^{-\gamma} ds - \kappa L \int_0^{t_1} (t_2 - s)^{\alpha-1} s^{-\gamma} ds. \end{aligned}$$

We have

$$\begin{aligned} |t_2^\alpha - t_1^\alpha| &\leq |t_2 - t_1|^\alpha, & 0 < \alpha \leq 1, \\ &\leq \alpha T^{\alpha-1} |t_2 - t_1|, & \alpha > 1, t_1, t_2 \in [0, T]. \end{aligned}$$

Hence, putting $M'_0 = M_0 \max\{1, \alpha T^{\alpha-1}\}$, $a = \min\{\alpha, 1\}$ we obtain

$$\begin{aligned} |J_1| &\leq \frac{M'_0}{\alpha_0} (t_2 - t_1)^a + \kappa L (t_1^{\alpha-\gamma} - t_2^{\alpha-\gamma}) B(\alpha, 1 - \gamma) + \kappa L \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} s^{-\gamma} ds \\ &\leq \frac{M'_0}{\alpha_0} (t_2 - t_1)^a + |\alpha - \gamma| \kappa L |t_2 - t_1| \delta^{\alpha-\gamma-1} + \frac{\kappa L \delta^{-\gamma}}{\alpha} (t_2 - t_1)^\alpha \\ &\leq \frac{M'_0}{\alpha_0} \xi^a + |\alpha - \gamma| \kappa L \xi \delta^{\alpha-\gamma-1} + \frac{\kappa L \delta^{-\gamma}}{\alpha} \xi^\alpha. \end{aligned}$$

Similarly, we obtain

$$|J_3| \leq \frac{M_0}{\alpha_0} (t_2 - t_1)^\alpha + \kappa L \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} s^{-\gamma} ds \leq \left(\frac{M_0 + \kappa L \delta^{-\gamma}}{\alpha_0} \right) \xi^\alpha.$$

To estimate J_2 , we note that $\|v(s)\| \leq \delta^{-\gamma} \|v\|_{C_\gamma(T, \mathbb{X})} \leq \delta^{-\gamma} L$ for $s \geq \delta$, so we have

$$|J_2| \leq \frac{T^\alpha}{\alpha \delta} \omega_K(\xi, \delta^{-\gamma} L). \quad (3.9)$$

From these estimates, we obtain that $B(\alpha_0, \alpha_1, P, L)$ is equicontinuous in $C([\delta, T]; \mathbb{X})$ for every $\delta > 0$. Using the Arzela-Ascoli theorem, we deduce that $B(\alpha_0, \alpha_1, P, L)$ is precompact in $C([\delta, T]; \mathbb{X})$ for every $\delta > 0$.

Now, we prove $B(\alpha_0, \alpha_1, P, L)$ is compact in $C_\gamma(T, \mathbb{X})$ by the diagonal argument. Since $B(\alpha_0, \alpha_1, P, L)$ is compact in $C\left(\left[\frac{T}{2}, T\right]; \mathbb{X}\right)$, we can find a sequence $A_{\alpha_1, n} w_{1, n}$ and a function $v_1 \in C\left(\left[\frac{T}{2}, T\right]; \mathbb{X}\right)$ such that

$$A_{\alpha_1, n, z_{1, n}} w_{1, n} \rightarrow v_1 \text{ in } C\left(\left[\frac{T}{2}, T\right]; \mathbb{X}\right).$$

In the sequence $A_{\alpha_1, n} w_{1, n}$ we can find a subsequence $A_{\alpha_2, n} w_{2, n}$ and a function $v_2 \in C\left(\left[\frac{T}{2^2}, T\right]; \mathbb{X}\right)$ such that

$$A_{\alpha_2, n, z_{2, n}} w_{2, n} \rightarrow v_2 \text{ in } C\left(\left[\frac{T}{2^2}, T\right]; \mathbb{X}\right).$$

We note that

$$A_{\alpha_2, n, \beta_{2, n}} w_{2, n} \rightarrow v_1 \text{ in } C\left(\left[\frac{T}{2}, T\right]; \mathbb{X}\right).$$

So we have $v_2|_{[T/2, T]} = v_1$. By induction we can construct sequences $A_{\alpha_k, n, z_{k, n}} w_{k, n}$ and the function v_k defined on $\left[\frac{T}{2^k}, T\right]$ such that

(i) $A_{\alpha_k, n, z_{k, n}} w_{k, n}$ is a subsequence of $A_{\alpha_{k-1}, n, z_{k-1}, n} w_{k-1, n}$.

(ii) $A_{\alpha_k, n, z_{k, n}} w_{k, n} \rightarrow v_k$ in $C\left(\left[\frac{T}{2^k}, T\right]; \mathbb{X}\right)$.

From the very last properties, we deduce that $v_k \Big|_{\left[\frac{T}{2^{k-1}}, T\right]} = v_{k-1}$. Therefore, we can define function $v(t) = v_k(t)$ for every $\left[\frac{T}{2^k}, T\right]$ in a unique way. We prove that $v \in C_\gamma(T, \mathbb{X})$ and $A_{\alpha_k, k, z_{k, k}} w_{k, k} \rightarrow v$ in $C_\gamma(T, \mathbb{X})$. Since $v_k \in C\left(\left[\frac{T}{2^k}, T\right]; \mathbb{X}\right)$, $k = 1, 2, \dots$, we obtain $v \in C((0, T]; \mathbb{X})$. On the other hand, the inequality (3.8) gives

$$t^\gamma \|A_{\alpha_k, k, z_{k, k}} w_{k, k}(t)\| \leq \max\{t^{\alpha_0}, t^{\alpha_1}\} \left(\frac{M_0}{\alpha_0} T^\gamma + \kappa L \max_{\alpha_0 \leq \alpha \leq \alpha_1} B(\alpha, 1 - \gamma) \right).$$

Letting $k \rightarrow \infty$ in the last inequality we obtain

$$t^\gamma \|v(t)\| \leq \max\{t^{\alpha_0}, t^{\alpha_1}\} \left(\frac{M_0}{\alpha_0} T^\gamma + \kappa L \max_{\alpha_0 \leq \alpha \leq \alpha_1} B(\alpha, 1 - \gamma) \right). \quad (3.10)$$

Hence $v \in C_\gamma(T, \mathbb{X})$. We now have to verify that $A_{\alpha_k, k, z_{k, k}} w_{k, k} \rightarrow v$ in $C_\gamma(T, \mathbb{X})$. For $\delta > 0$, we have

$$\|A_{\alpha_k, k, z_{k, k}} w_{k, k} - v\|_{C_\gamma(T, \mathbb{X})} \leq L_1 + L_2$$

where

$$\begin{aligned} L_1 &= \max_{0 < t \leq \delta} t^\gamma |A_{\alpha_k, k, z_{k, k}} w_{k, k}(t) - v(t)|, \\ L_2 &= \max_{\delta < t \leq T} t^\gamma |A_{\alpha_k, k, z_{k, k}} w_{k, k}(t) - v(t)|. \end{aligned}$$

For $0 < \delta < \min\{1, T\}$, in view of (3.8) and (3.10) we have

$$|L_1| \leq \delta^{\alpha_0} \left(\frac{M_0}{\alpha_0} T^\gamma + \kappa L \max_{\alpha_0 \leq \alpha \leq \alpha_1} B(\alpha, 1 - \gamma) \right).$$

Since $A_{\alpha_k, k, z_{k, k}} w_{k, k} \rightarrow v$ in $C([\delta, T]; \mathbb{X})$ as $k \rightarrow \infty$, we get

$$\limsup_{k \rightarrow \infty} \|A_{\alpha_k, k, z_{k, k}} w_{k, k} - v\|_{C_\gamma(T, \mathbb{X})} \leq \delta^{\alpha_0} \left(\frac{M_0}{\alpha_0} T^\gamma + \kappa L \max_{\alpha_0 \leq \alpha \leq \alpha_1} B(\alpha, 1 - \gamma) \right)$$

for every $\delta \in (0, \min\{1, T\})$. Let $\delta \rightarrow 0^+$, we obtain $\limsup_{k \rightarrow \infty} \|A_{\alpha_k, k, z_{k, k}} w_{k, k} - v\|_{C_\gamma(T, \mathbb{X})} = 0$. This completes the proof of part (a).

Proof of (b): Since $w_n \rightarrow w$ in $C_\gamma(T, \mathbb{X})$, there exists a constant L such that $\|w\|_{C_\gamma(T, \mathbb{X})}, \|w_n\|_{C_\gamma(T, \mathbb{X})} \leq L$ for every $n = 1, 2, \dots$. Choose $\alpha_0, \alpha_1 \in (0, 1)$ such that $\alpha_0 \leq a_n \leq \alpha_1$ for every $n = 1, 2, \dots$. Hence we have $A_{a_n, z_n} w_n \in B(\alpha_0, \alpha_1, P, L)$. Assume that $A_{a_n, z_n} w_n \not\rightarrow A_{\alpha, \beta} w$ in $C_\gamma(T, \mathbb{X})$ as $n \rightarrow \infty$. From Lemma 3.2, we can find an $\epsilon_0 > 0$ and a subsequence of $(A_{a_n, z_n} w_n)$, still denote by the same sequence, which converges to $z \in C_\gamma(T, \mathbb{X})$ and $\|A_{a_n, z_n} w_n - A_{\alpha, \beta} w\|_{C_\gamma(T, \mathbb{X})} \geq \epsilon_0 > 0$. Let $n \rightarrow \infty$ we obtain $\|z - A_{\alpha, z} w\|_{C_\gamma(T, \mathbb{X})} \geq \epsilon_0 > 0$. We claim that $z = A_{\alpha, z} w$ which is a contradiction. We have

$$\begin{aligned} A_{a_n, z_n} w_n(t) &= \int_0^t \frac{K(t, s, a_n, z_n, w_n(s))}{(t - s)^{1 - a_n}} ds \\ &= t^{a_n} \int_0^1 \frac{K(t, \theta t, a_n, z_n, w_n(\theta t))}{(1 - \theta)^{1 - a_n}} d\theta. \end{aligned}$$

Fixing $t > 0$, we put

$$F_n(\theta) = \frac{K(t, \theta t, a_n, z_n, w_n(\theta t))}{(1 - \theta)^{1-a_n}}, \quad F(\theta) = \frac{K(t, \theta t, \alpha, z, w(\theta t))}{(1 - \theta)^{1-\alpha}}.$$

We have

$$\begin{aligned} \|F_n(\theta) - F(\theta)\| &\leq \frac{M_0 + \kappa \|w_n(\theta t)\|}{(1 - \theta)^{1-a_n}} + \frac{M_0 + \kappa \|w(\theta t)\|}{(1 - \theta)^{1-\alpha}} \\ &\leq \frac{2M_0 + 2\kappa(\theta t)^{-\gamma} L}{(1 - \theta)^{1-a_0}} := g(\theta). \end{aligned}$$

Since g is in $L^1(0, T)$ and that $\lim_{n \rightarrow \infty} \|F_n(\theta) - F(\theta)\| = 0$, we can apply the dominated convergence theorem of Lebesgue to obtain $\lim_{n \rightarrow \infty} \|\int_0^1 (F_n(\theta) - F(\theta)) d\theta\| = 0$. It follows that $z(t) = \lim_{n \rightarrow \infty} A_{a_n, z_n} w_n(t) = A_{\alpha, z} w(t)$. \square

Now, we state and prove the continuity of the solutions of **the general nonlinear Abel integral of the second kind** with respect to the fractional parameter α .

Theorem 3.2. *Suppose that the assumptions of Theorem 3.1 and Lemma 3.2 hold. Let $0 < \alpha_0 < \alpha_1$, $\gamma \in [0, 1)$, $\nu, \mu_1, \dots, \mu_k \in (0, 1]$, $\kappa_0, \kappa > 0$, $(\alpha, z), (\alpha', z') \in [\alpha_0, \alpha_1] \times P$.*

(a) *If $g_{\alpha, z} \in C_\gamma(T, \mathbb{X})$ for all $(\alpha, z) \in [\alpha_0, \alpha_1] \times P$ and*

$$\lim_{(\alpha', z') \rightarrow (\alpha, z)} \|g_{\alpha', z'} - g_{\alpha, z}\|_{C_\gamma(T, \mathbb{X})} = 0.$$

then the nonlinear equation (3.4) has a unique solution and we have

$$\lim_{(\alpha', z') \rightarrow (\alpha, z)} \|u_{\alpha', z'} - u_{\alpha, z}\|_{C_\gamma(T, \mathbb{X})} = 0.$$

(b) *If $g_{\alpha, z} \in L^p(0, T; \mathbb{X})$ for all $(\alpha, z) \in [\alpha_0, \alpha_1] \times P$ and*

$$\lim_{(\alpha', z') \rightarrow (\alpha, z)} \|g_{\alpha', z'} - g_{\alpha, z}\|_{L^p(0, T; \mathbb{X})} = 0.$$

Then

$$\lim_{(\alpha', z') \rightarrow (\alpha, z)} \|u_{\alpha', z'} - u_{\alpha, z}\|_{L^p(0, T; \mathbb{X})} = 0.$$

(c) *Letting $z = (\beta_1, \dots, \beta_k)$, $z' = (\beta'_1, \dots, \beta'_k) \in P$, $\mu = (\mu_1, \dots, \mu_k)$, $\mu_j \in (0, 1]$, $j = 1, \dots, k$, we assume that*

$$\begin{aligned} \|g_{\alpha', z'} - g_{\alpha, z}\|_{C_\gamma(T, \mathbb{X})} &\leq \kappa_0(|\alpha' - \alpha|^\nu + |z' - z|^\mu), \\ \|K(t, s, \alpha', z', w) - K(t, s, \alpha, z, w)\| &\leq \kappa(|\alpha' - \alpha|^\nu + |z' - z|^\mu)(\|w\| + 1) \end{aligned}$$

where $0 \leq s \leq t \leq T$, $|z' - z|^\mu := \sum_{j=1}^k |\beta'_j - \beta_j|^{\mu_j}$. Then there is a $C = C(\alpha_0, \alpha_1, P)$ such that

$$\|u_{\alpha', z'} - u_{\alpha, z}\|_{C_\gamma(T, \mathbb{X})} \leq C_0(|\alpha' - \alpha|^\nu + |z' - z|^\mu)$$

where

$$C_0 = C(\kappa_0 + \kappa + 1)^2 E_{\alpha_0, 1-\gamma}^2 (\kappa \Gamma(\alpha_1) \max\{T^{\alpha_0}, T^{\alpha_1}\})(1 + \|g_{\alpha, z}\|_{C_\gamma(T, \mathbb{X})}).$$

In addition, let $\lambda \geq \nu$, $\rho_j \geq \mu_j$, $j \in \overline{1, k}$ and let (a_n, z_n) be random variables satisfying $(a_n, z_n) \in [\alpha_0, \alpha_1] \times P$, $z_n = (z_{1n}, \dots, z_{kn})$. Then

$$\mathbb{E} \|u_{a_n, z_n} - u_{\alpha, z}\|_{C_\gamma(T, \mathbb{X})} \leq C_0 \left((\mathbb{E} |a_n - \alpha|^\lambda)^{\nu/\lambda} + \sum_{j=1}^k (\mathbb{E} |z_{jn} - \beta_j|^{\rho_j})^{\mu_j/\rho_j} \right).$$

(d) If K is as in part (c) and

$$\|g_{\alpha', z'} - g_{\alpha, z}\|_{L^p(0, T; \mathbb{X})} \leq \kappa_0(|\alpha' - \alpha|^\nu + |z' - z|^\mu),$$

then we have

$$\|u_{\alpha', z'} - u_{\alpha, z}\|_{L^p(0, T; \mathbb{X})} \leq C_0(|\alpha' - \alpha|^\nu + |z' - z|^\mu)$$

for a constant

$$C_0 = C(\kappa_0 + \kappa + 1)^2 E_{\alpha_0, \alpha_0}^2 (\kappa \Gamma(\alpha_1) \max\{T^{\alpha_0}, T^{\alpha_1}\}) (\|g_{\alpha, z}\|_{L^p(0, T; \mathbb{X})} + 1).$$

In addition, let (a_n, z_n) be random variables as in (c). Then

$$\mathbb{E} \|u_{a_n, z_n} - u_{\alpha, z}\|_{L^p(0, T; \mathbb{X})} \leq C_0 \left((\mathbb{E}|a_n - \alpha|^\lambda)^{\nu/\lambda} + \sum_{j=1}^k (\mathbb{E}|z_{jn} - \beta_j|^{\rho_j})^{\mu_j/\rho_j} \right).$$

(e) Assume that $K(t, s, \alpha, z, w)$ has the derivative $\frac{\partial K}{\partial \alpha}$ and the Frechet derivative DK with respect to the variable w . Moreover, we assume that $\frac{\partial g}{\partial \alpha}(\cdot, \alpha) \in C_\gamma(T, \mathbb{X})$ with $0 < \gamma \leq 1$, $\frac{\partial K}{\partial \alpha} \in C(\Delta_T \times \mathbb{X}; \mathbb{X})$ and there exists $\omega : [0, 1] \rightarrow \mathbb{R}$ such that $\lim_{\delta \rightarrow 0} \omega(\delta) = 0$ and

$$\omega_{DK^*}(w, \delta) \leq \omega(\delta) \|w\|$$

where

$$\omega_{DK^*}(w, \delta) = \sup_{B_1} \|DK^*(t_2, s, \alpha, z, w) - DK^*(t_1, s, \alpha, z, w)\|$$

and

$$B_1 = \{(t_2, t_1, s, \alpha, z) : 0 \leq s \leq t_1 \leq t_2, |t_2 - t_1| \leq \delta, \alpha \in [\alpha_0, \alpha_1], z \in P\}.$$

Then $u_{\alpha, z}$ is differentiable with respect to α and

$$\left\| \frac{\partial u_{\alpha, z}}{\partial \alpha} \right\|_{C_\gamma(T, \mathbb{X})} \leq \Gamma(1 - \gamma) E_{\alpha, 1-\gamma}(MT^\alpha) \|g_{1\alpha, z}\|_{C_\gamma(T, \mathbb{X})},$$

where

$$\begin{aligned} g_{1\alpha, z}(t) &= \frac{\partial g_{\alpha, z}^*}{\partial \alpha} + \int_0^t \frac{\partial K^*}{\partial \alpha}(t, s, \alpha, z, u_{\alpha, z}(s)) \frac{ds}{(t-s)^{1-\alpha}} \\ &\quad + \int_0^t \frac{K^*(t, s, \alpha, u_{\alpha, z}(s)) \ln(t-s)}{(t-s)^{1-\alpha}} ds. \end{aligned}$$

Proof. Proof of (a): Assume that $\|u_{\alpha', z'} - u_{\alpha, z}\|_{C_\gamma(T, \mathbb{X})} \not\rightarrow 0$ as $\alpha' \rightarrow \alpha$. We can choose an $\epsilon_0 > 0$ and a sequence u_{a_n, z_n} , such that

$$\lim_{n \rightarrow \infty} (a_n, z_n) = (\alpha, z) \text{ and } \|u_{a_n, z_n} - u_{\alpha, z}\|_{C_\gamma(T, \mathbb{X})} \geq \epsilon_0 > 0.$$

From the first part of Theorem 3.1, we can find a constant L such that $\|u_{a_n, z_n}\|_{C_\gamma(T, \mathbb{X})} \leq L$ for every $n \rightarrow \infty$. It follows that $A_{a_n, z_n}^* u_{a_n, z_n} \in B(\alpha_0, \alpha_1, P, L)$. So by Lemma 3.2 we can find a subsequence, still denoted by $A_{a_n, z_n}^* u_{a_n, z_n}$, and an element $x \in C_\gamma(T, \mathbb{X})$ such that $\lim_{n \rightarrow \infty} A_{a_n, z_n}^* u_{a_n, z_n} = x \in C_\gamma(T, \mathbb{X})$. It follows that

$$u_{a_n, z_n} = g_{a_n, z_n}^* + A_{a_n, z_n}^* u_{a_n, z_n} \rightarrow g_{\alpha, z}^* + x := u.$$

From Lemma 3.2, we obtain $A_{a_n, z_n}^* u_{a_n, z_n} \rightarrow A_{\alpha, z}^* u$ as $n \rightarrow \infty$. Hence $u = g_{\alpha, z}^* + A_{\alpha, z}^* u$. But $u_{\alpha, z} = g_{\alpha, z}^* + A_{\alpha, z}^* u_{\alpha, z}$, hence, by the uniqueness we have $u = u_{\alpha, z}$ and $u_{a_n, z_n} \rightarrow u_{\alpha, z}$ in $C_\gamma(T, \mathbb{X})$. This contradicts with the assumption $\|u_{a_n, z_n} - u_{\alpha, z}\|_{C_\gamma(T, \mathbb{X})} \geq \epsilon_0 > 0$.

Proof of (b): Choose $\varphi \in C_c^\infty(-1, 1)$, $|\varphi(t)| \leq 1$ for $t \in [-1, 1]$. Let $\delta > 0$ and $\varphi_\delta(t) = \frac{1}{C_\varphi \delta} \varphi(\frac{t}{\delta})$ with $C_\varphi = \int_{-1}^1 \varphi(s) ds$. We approximate $g_{\alpha, z}^*$ by $G_{\delta, \alpha, z} := \varphi_\delta * g_{\alpha, z}^* \in C([0, T]; \mathbb{X})$. Let $v_{\delta, \alpha, z}$ be the solution of

$$v_{\delta, \alpha, z} = G_{\delta, \alpha, z} + A_{\alpha, z}^* v_{\delta, \alpha, z}.$$

We can use the triangle inequality to get

$$\begin{aligned} \|u_{\alpha', z'} - u_{\alpha, z}\|_{L^p(0, T; \mathbb{X})} &\leq \|u_{\alpha', z'} - v_{\delta, \alpha', z'}\|_{L^p(0, T; \mathbb{X})} + \|v_{\delta, \alpha', z'} - v_{\delta, \alpha, z}\|_{L^p(0, T; \mathbb{X})} \\ &\quad + \|v_{\delta, \alpha, z} - u_{\alpha, z}\|_{L^p(0, T; \mathbb{X})}. \end{aligned}$$

Calculating directly, we obtain

$$\begin{aligned} \|G_{\delta, \alpha, z} - g_{\alpha, z}\|_{L^p(0, T; \mathbb{X})} &\leq C \sup_{|\xi| \leq \delta} \|g_{\alpha, z}(\cdot + \xi) - g_{\alpha, z}\|_{L^p(0, T; \mathbb{X})}, \\ \|G_{\delta, \alpha', z'} - G_{\delta, \alpha, z}\|_{L^p(0, T; \mathbb{X})} &\leq \|g_{\alpha', z'} - g_{\alpha, z}\|_{L^p(0, T; \mathbb{X})}. \end{aligned}$$

From the last two inequalities, Lemma 3.1 (e) gives

$$\begin{aligned} \|v_{\delta, \alpha, z} - u_{\alpha, z}\|_{L^p(0, T; \mathbb{X})} &\leq C(\alpha_0, \alpha_1) C \sup_{|\xi| \leq \delta} \|g_{\alpha, z}(\cdot + \xi) - g_{\alpha, z}\|_{L^p(0, T; \mathbb{X})}, \\ \|u_{\alpha', z'} - v_{\delta, \alpha', z'}\|_{L^p(0, T; \mathbb{X})} &\leq C(\alpha_0, \alpha_1) C \sup_{|\xi| \leq \delta} \|g_{\alpha', z'}(\cdot + \xi) - g_{\alpha', z'}\|_{L^p(0, T; \mathbb{X})}, \end{aligned}$$

where

$$C(\alpha_0, \alpha_1) = 1 + \kappa \Gamma(\alpha_1) T E_{\alpha_0, \alpha_0}(\kappa \Gamma(\alpha_1) \max\{T^{\alpha_0}, T^{\alpha_1}\}).$$

Estimating directly gives

$$\begin{aligned} \|g_{\alpha', z'}(\cdot + \xi) - g_{\alpha', z'}\|_{L^p(0, T; \mathbb{X})} &\leq \|g_{\alpha', z'} - g_{\alpha, z}\|_{L^p(0, T; \mathbb{X})} + \|g_{\alpha', z'}(\cdot + \xi) - g_{\alpha, z}(\cdot + \xi)\|_{L^p(0, T; \mathbb{X})} \\ &\quad + \|g_{\alpha, z}(\cdot + \xi) - g_{\alpha, z}\|_{L^p(0, T; \mathbb{X})} \\ &\leq 2\|g_{\alpha', z'} - g_{\alpha, z}\|_{L^p(0, T; \mathbb{X})} + \|g_{\alpha, z}(\cdot + \xi) - g_{\alpha, z}\|_{L^p(0, T; \mathbb{X})}. \end{aligned}$$

Hence

$$\begin{aligned} \|u_{\alpha', z'} - u_{\alpha, z}\|_{L^p(0, T; \mathbb{X})} &\leq C(\alpha_0, \alpha_1) C \sup_{|\xi| \leq \delta} \|g_{\alpha, z}(\cdot + \xi) - g_{\alpha, z}\|_{L^p(0, T; \mathbb{X})} + \\ &\quad 2C(\alpha_0, \alpha_1) \|g_{\alpha', z'} - g_{\alpha, z}\|_{L^p(0, T; \mathbb{X})} + \|v_{\delta, \alpha', z'} - v_{\delta, \alpha, z}\|_{L^p(0, T; \mathbb{X})}. \end{aligned}$$

Now, choosing a sequence $(\alpha'_n, z'_n) \rightarrow (\alpha, z)$ as $n \rightarrow \infty$ and using Part (a), we can obtain

$$\limsup_{n \rightarrow \infty} \|u_{\alpha'_n, z'_n} - u_{\alpha, z}\|_{L^p(0, T; \mathbb{X})} \leq C(\alpha_0, \alpha_1) C \sup_{|\xi| \leq \delta} \|g_{\alpha, z}(\cdot + \xi) - g_{\alpha, z}\|_{L^p(0, T; \mathbb{X})}.$$

Letting $\delta \rightarrow 0^+$ gives

$$\limsup_{n \rightarrow \infty} \|u_{\alpha'_n, z'_n} - u_{\alpha, z}\|_{L^p(0, T; \mathbb{X})} = 0.$$

This completes the proof of Part (b).

Proof of (c): We have

$$\begin{aligned} u_{\alpha', z'}(t) - u_{\alpha, z}(t) &= g_{\alpha', z'}^* - g_{\alpha, z}^* + I_1(\alpha', \alpha, z', z) + I_2(\alpha', \alpha, z', z) \\ &\quad + \int_0^t \frac{K^*(t, s, \alpha, z, u_{\alpha', z'}(s)) - K^*(t, s, \alpha, z, u_{\alpha, z}(s))}{(t-s)^{1-\alpha}} ds \end{aligned} \quad (3.11)$$

where

$$\begin{aligned} I_1(\alpha', \alpha, z', z) &= \int_0^t K^*(t, s, \alpha', z', u_{\alpha', z'}(s))((t-s)^{\alpha'-1} - (t-s)^{\alpha-1}) ds, \\ I_2(\alpha', \alpha, z', z) &= \int_0^t \frac{K^*(t, s, \alpha', z', u_{\alpha', z'}(s)) - K^*(t, s, \alpha, z, u_{\alpha', z'}(s))}{(t-s)^{1-\alpha}} ds. \end{aligned}$$

Now, put $w = u_{\alpha', z'} - u_{\alpha, z}$, $G \equiv G(\alpha', \alpha, z', z) = g_{\alpha', z'}^* - g_{\alpha, z}^* + I_1(\alpha', \alpha, z', z) + I_2(\alpha', \alpha, z', z)$ and

$$K_1(t, s, w(s)) = K^*(t, s, \alpha, z, u_{\alpha, z}(s) + w(s)) - K^*(t, s, \alpha, z, u_{\alpha, z}(s)).$$

We can rewrite the equation (3.11) as

$$w(t) = G(t) + \int_0^t \frac{K_1(t, s, w(s)) ds}{(t-s)^{1-\alpha}}.$$

Using Part (a) of Theorem 3.1 we obtain

$$\|w\|_{C_\gamma(T, \mathbb{X})} \leq \Gamma(1-\gamma) E_{\alpha, 1-\gamma}(\kappa \Gamma(\alpha) T^\alpha) \|G\|_{C_\gamma(T, \mathbb{X})}. \quad (3.12)$$

We estimate $\|G\|_{C_\gamma(T, \mathbb{X})}$. From the definition of the term we have

$$\|G\|_{C_\gamma(T, \mathbb{X})} \leq \|g_{\alpha', z'}^* - g_{\alpha, z}^*\|_{C_\gamma(T, \mathbb{X})} + \|I_1(\alpha', \alpha, z', z)\|_{C_\gamma(T, \mathbb{X})} + \|I_2(\alpha', \alpha, z', z)\|_{C_\gamma(T, \mathbb{X})}.$$

For $\alpha < \alpha'$, applying directly Lemma 3.1, Part (a) gives

$$\begin{aligned} \|I_1(\alpha', \alpha, z', z)\| &\leq \int_0^t \kappa \|u_{\alpha', z'}(s)\| (t-s)^{\alpha-1} ((t-s)^{\alpha'-\alpha} - 1) ds \\ &\leq \kappa \|u_{\alpha', z'}\|_{C_\gamma(T, \mathbb{X})} \int_0^t s^{-\gamma} (t-s)^{\alpha-1} ((t-s)^{\alpha'-\alpha} - 1) ds \\ &\leq \kappa \|u_{\alpha', z'}\|_{C_\gamma(T, \mathbb{X})} C |\alpha' - \alpha| t^{\alpha-\gamma} (1 + |\ln t|). \end{aligned}$$

It follows that

$$\|I_1(\alpha', \alpha, z', z)\|_{C_\gamma(T, \mathbb{X})} \leq C \kappa \|u_{\alpha', z'}\|_{C_\gamma(T, \mathbb{X})} |\alpha' - \alpha|.$$

The same inequality hold for $\alpha > \alpha'$. Next we estimate $I_2(\alpha', \alpha, z', z)$ as follows

$$\begin{aligned} \|I_2(\alpha', \alpha, z', z)\| &\leq \int_0^t \kappa (|\alpha' - \alpha|^\nu + |z' - z|^\mu) (\|u_{\alpha', z'}(s)\| + 1) (t-s)^{1-\alpha} ds \\ &\leq \kappa (\|u_{\alpha', z'}\|_{C_\gamma(T, \mathbb{X})} + 1) C (|\alpha' - \alpha|^\nu + |z' - z|^\mu) t^{\alpha-\gamma} (1 + |\ln t|). \end{aligned}$$

Hence

$$\|I_2(\alpha', \alpha, z', z)\|_{C_\gamma(T, \mathbb{X})} \leq C \kappa (\|u_{\alpha', z'}\|_{C_\gamma(T, \mathbb{X})} + 1) (|\alpha' - \alpha|^\nu + |z' - z|^\mu).$$

Finally, we can estimate similarly to obtain

$$\|g_{\alpha', z'}^* - g_{\alpha, z}^*\|_{C_\gamma(T, \mathbb{X})} \leq C(\kappa_0 + \kappa) (|\alpha' - \alpha|^\nu + |z' - z|^\mu).$$

Substituting these estimates for $I_1(\alpha', \alpha, z', z)$, $I_2(\alpha', \alpha, z', z)$, $g_{\alpha', z'}^* - g_{\alpha, z}^*$ into (3.12) gives

$$\|w\|_{C_\gamma(T, \mathbb{X})} \leq C(\kappa_0 + \kappa)E_{\alpha, 1-\gamma}(\kappa\Gamma(\alpha)T^\alpha)(\|u_{\alpha', z'}\|_{C_\gamma(T, \mathbb{X})} + 1)(|\alpha' - \alpha|^\nu + |z' - z|^\mu).$$

Using the part (a) of Theorem 3.1, we obtain

$$\begin{aligned} \|w\|_{C_\gamma(T, \mathbb{X})} &\leq C(\kappa_0 + \kappa + 1)^2 E_{\alpha, 1-\gamma}^2(\kappa\Gamma(\alpha)T^\alpha)(|\alpha' - \alpha|^\nu + |z' - z|^\mu) \\ &\leq C_0(|\alpha' - \alpha|^\nu + |z' - z|^\mu) \end{aligned}$$

where

$$C_0 = C(\kappa_0 + \kappa + 1)^2 E_{\alpha_0, 1-\gamma}^2(\kappa\Gamma(\alpha_1) \max\{T^{\alpha_0}, T^{\alpha_1}\})(1 + \|g_{\alpha, z}\|_{C_\gamma(T, \mathbb{X})}).$$

We turn to the case of random order. Using the last inequality and the Jensen's inequality $\mathbb{E}|X| \leq (\mathbb{E}|X|^p)^{1/p}$ for $p \geq 1$, we have

$$\begin{aligned} \mathbb{E}\|u_{a_n, z_n} - u_{\alpha, z}\|_{C_\gamma(T, \mathbb{X})} &\leq C_0 \mathbb{E}(|a_n - \alpha|^\nu + |z_n - z|^\mu) \\ &\leq C_0 \left(\mathbb{E}(|a_n - \alpha|^\lambda)^{\nu/\lambda} + C_0 \sum_{j=1}^k (\mathbb{E}|z_{jn} - \beta_j|^{\rho_j})^{\mu_j/\rho_j} \right). \end{aligned}$$

Proof of (d): We have

$$\begin{aligned} I_1(\alpha', \alpha, z', z) &= \int_0^t K^*(t, s, \alpha', z', u_{\alpha', z'}(s))((t-s)^{\alpha'-1} - (t-s)^{\alpha-1})ds, \\ I_2(\alpha', \alpha, z', z) &= \int_0^t \frac{K^*(t, s, \alpha', z', u_{\alpha', z'}(s)) - K^*(t, s, \alpha, z, u_{\alpha, z}(s))}{(t-s)^{1-\alpha}} ds. \end{aligned}$$

For $\alpha < \alpha'$, applying directly Lemma 3.1 gives

$$\begin{aligned} \|I_1(\alpha', \alpha, z', z)\|_{L^p(0, T; \mathbb{X})} &\leq \kappa \|u_{\alpha', z'}\|_{L^p(0, T; \mathbb{X})} \int_0^t (t-s)^{\alpha-1} ((t-s)^{\alpha'-\alpha} - 1) ds \\ &\leq \kappa \|u_{\alpha', z'}\|_{L^p(0, T; \mathbb{X})} C |\alpha' - \alpha| t^\alpha (1 + |\ln t|). \end{aligned}$$

The same inequality hold for $\alpha > \alpha'$. Now we estimate $I_2(\alpha', \alpha, z', z)$.

$$\|I_2(\alpha', \alpha, z', z)\|_{L^p(0, T; \mathbb{X})} \leq \kappa \|u_{\alpha', z'}\|_{L^p(0, T; \mathbb{X})} C(|\alpha' - \alpha|^\nu + |z' - z|^\mu) t^\alpha (1 + |\ln t|).$$

Now, put $w = u_{\alpha', z'} - u_{\alpha, z}$, $G \equiv G(\alpha', \alpha, z', z) = g_{\alpha', z'}^* - g_{\alpha, z}^* + I_1(\alpha', \alpha, z', z) + I_2(\alpha', \alpha, z', z)$ and

$$K_1(t, s, w(s)) = K^*(t, s, \alpha, z, u_{\alpha, z}(s) + w(s)) - K^*(t, s, \alpha, z, u_{\alpha, z}(s)). \quad (3.13)$$

We can rewrite the equation (3.11) as

$$w(t) = G(t) + \int_0^t \frac{K_1(t, s, w(s)) ds}{(t-s)^{1-\alpha}}.$$

Using Theorem 3.1, we obtain

$$\|w\|_{L^p(0, T; \mathbb{X})} \leq (1 + \kappa\Gamma(\alpha)\alpha^{-1}T^\alpha E_{\alpha, \alpha}(\kappa\Gamma(\alpha)T^\alpha)) \|G\|_{L^p(0, T; \mathbb{X})}.$$

From the estimate of G we complete the proof of the first part of (d). The second part can be verified as in the proof of previous part.

Proof of (e): We consider the equation

$$w_{\alpha,z}(t) = g_1(t) + \int_0^t \frac{DK(t, s, \alpha, z, u_{\alpha,z})w_{\alpha,z}(s)}{(t-s)^{1-\alpha}} ds. \quad (3.14)$$

The equation (3.14) can be seen as the "derivative" of the linear Abel equation (3.15). Using the first part of the proof we deduce that the equation (3.14) has a unique solution $w_{\alpha,z} \in C_\gamma(T, \mathbb{X})$.

Now, we claim that $\frac{\partial u_{\alpha,z}}{\partial \alpha} = w_{\alpha,z}$.

Put $w_{\alpha,h} = \frac{u_{\alpha+h,z} - u_{\alpha,z}}{h}$ we obtain

$$w_{\alpha,h}(t) = h^{-1}G(\alpha + h, \alpha, z', z)(t) + \int_0^t \frac{K_1(t, s, hw_{\alpha,h}(s))}{h(t-s)^{1-\alpha}} ds$$

where $K_1(t, s, \cdot)$ is defined in (3.13). We note that $h^{-1}G(\alpha + h, \alpha, z', z) \rightarrow g_1$ in $C_\gamma(T, \mathbb{X})$. From the Lipschitz property of $K(t, s, \alpha, z, w)$ with respect to the variable w , we obtain

$$\|K_1(t, s, w)\| \leq \kappa \|w\|.$$

Therefore, we have

$$\|w_{\alpha,h}\|_{C_\gamma(T, \mathbb{X})} \leq \|h^{-1}G(\alpha + h, \alpha, z', z)\|_{C_\gamma(T, \mathbb{X})} \Gamma(1-\gamma) E_{\alpha, 1-\gamma}(M_1 T^\alpha) := M_2.$$

We verify the equicontinuity of K_1 with respect to the variable t . We first have

$$\begin{aligned} h^{-1}K_1(t, s, hw(s)) &= h^{-1}(K^*(t, s, \alpha, z, u_{\alpha,z}(s) + hw(s)) - K^*(t, s, \alpha, z, u_{\alpha,z}(s))) \\ &= \int_0^1 DK^*(t, s, \alpha, z, u_{\alpha,z}(s) + h\theta w(s))w(s) d\theta. \end{aligned}$$

So we have

$$\begin{aligned} &\|h^{-1}K_1(t_2, s, hw(s)) - h^{-1}K_1(t_1, s, hw(s))\| \\ &\leq \int_0^1 \|DK^*(t_2, s, \alpha, z, u_{\alpha,z}(s) + h\theta w(s)) - DK^*(t_1, s, \alpha, z, u_{\alpha,z}(s) + h\theta w(s))w(s)\| ds \\ &\leq \omega(|t_2 - t_1|) \|u_{\alpha,z}(s) + h\theta w(s)\|. \end{aligned}$$

So we have the equi-continuity with respect to t of the family $h^{-1}K_1(t, s, hw(s))$. As in Part (a), we can use a compactness argument to prove that $w_{\alpha,h} \rightarrow w_{\alpha,z}$ in $C_\gamma(T, \mathbb{X})$. This completes the proof of our theorem. \square

3.2 Continuity of the solutions of fractional equations with sequential derivatives

We start by applying the previous results for the general equation (3.1). For convenience, we recall that if $\mathcal{D}_t^{\sigma_k} y(t) = \psi(t)$ then we can rewrite the equation (3.1) as the integral equation (3.2).

Theorem 3.3. *Let $\eta_0 \in (0, 1)$ and $\eta_0 \leq \eta_j, \eta'_j \leq 1$, $0 < B_0 < B_1$ and put $z = (\eta_1, \dots, \eta_k)$, $z' = (\eta'_1, \dots, \eta'_k)$, $\sigma_k = \sum_{j=1}^k \eta_k$, $\sigma'_k = \sum_{j=1}^k \eta'_k$. Assume that $p_j \in C[0, T]$, $j = 1, \dots, k$, $b_j \in (B_0, B_1)$. We denote by ψ_z the solution of (3.2) and by y_z the solution of (3.1).*

- (a) If $\gamma \in (1 - \eta_0, 1)$, $f \in C_\gamma(T; \mathbb{X})$, then equation (3.2) has a unique solution $\psi_z \in C_\gamma(T; \mathbb{X})$ and the equation (3.1) has a unique solution $y_z \in C_\gamma(T; \mathbb{X})$ such that $\mathcal{D}_t^{\sigma_k} y_z(t) = \psi(t)$. Moreover, there exists a constant $C = C(\eta_0, B_0, B_1)$ such that

$$\|\mathcal{D}_t^{\sigma'_k} y_{z'} - \mathcal{D}_t^{\sigma_k} y_z\|_{C_\gamma(T; \mathbb{X})} \leq C \left(1 + \|f\|_{C_\gamma(T; \mathbb{X})} + \sum_{j=1}^k |b_j| \right) \sum_{j=1}^k |\eta'_j - \eta_j|.$$

In addition, let $\rho_j \geq 1$, $j = 1, \dots, k$. If η_{jn} are random, and $\eta_{jn} \in [\eta_0, 1]$, $z_n = (\eta_{1n}, \dots, \eta_{kn})$, $\sigma_{kn} = \sum_{j=1}^k \eta_{jn}$ then

$$\mathbb{E} \|\mathcal{D}_t^{\sigma_{kn}} y_{z_n} - \mathcal{D}_t^{\sigma_k} y_z\|_{C_\gamma(T; \mathbb{X})} \leq C \left(1 + \|f\|_{C_\gamma(T; \mathbb{X})} + \sum_{j=1}^k |b_j| \right) \sum_{j=1}^k (\mathbb{E} |\eta_{jn} - \eta_j|^{\rho_j})^{1/\rho_j}.$$

- (b) If $p \in \left[1, \frac{1}{1-\eta_0}\right)$, $f \in L^p(0, T; \mathbb{X})$ then

$$\|\mathcal{D}_t^{\sigma_k} y_{z'} - \mathcal{D}_t^{\sigma'_k} y_z\|_{L^p(0, T; \mathbb{X})} \leq C \left(1 + \|f\|_{L^p(0, T; \mathbb{X})} + \sum_{j=1}^k |b_j| \right) \sum_{j=1}^k |\eta'_j - \eta_j|.$$

In addition, in the case of random order, we have

$$\mathbb{E} \|\mathcal{D}_t^{\sigma_{kn}} y_{z_n} - \mathcal{D}_t^{\sigma_k} y_z\|_{L^p(0, T; \mathbb{X})} \leq C \left(1 + \|f\|_{L^p(0, T; \mathbb{X})} + \sum_{j=1}^k |b_j| \right) \sum_{j=1}^k (\mathbb{E} |\eta_{jn} - \eta_j|^{\rho_j})^{1/\rho_j}.$$

Proof. We choose $\mathbb{X} = \mathbb{R}$ and verify the conditions of Theorem 3.2, Part (c). In fact, we have $(k-1)\eta_0 \leq \sigma_k - \eta_k = \sigma_{k-1} \leq k-1$ and $(k-1-j)\eta_0 \leq \sigma_k - \sigma_j - \eta_k = \sigma_{k-1} - \sigma_j \leq k-1-j$ for every $0 \leq j \leq k-1$. This implies

$$0 \leq (t-s)^{\sigma_k - \eta_k}, (t-s)^{\sigma_k - \sigma_j - \eta_k} \leq \max\{1, T^{k-1}\} := T_k.$$

Hence

$$\begin{aligned} |K(t, s, z, v) - K(t, s, z, w)| &\leq \left| p_k(t) \frac{(t-s)^{\sigma_k - \eta_k}}{\Gamma(\sigma_k)} + \sum_{j=1}^{k-1} p_{k-j}(t) \frac{(t-s)^{\sigma_k - \sigma_j - \eta_k}}{\Gamma(\sigma_k)} \right| |v - w| \\ &\leq \frac{k M_p T_k}{\Gamma((k-1)\eta_0)} |v - w| \end{aligned}$$

where $M_p = \max_{1 \leq j \leq k} \|p_j\|_{C[0, T]}$. So, $K(t, s, z, w)$ is uniformly Lipschitz with respect to the variable $w \in \mathbb{X}$. We verify the Lipschitz condition with respect to $z = (\eta_1, \dots, \eta_k)$.

$$\begin{aligned} K(t, s, z', w) - K(t, s, z, w) &= p_k(t) \left(\frac{(t-s)^{\sigma'_k - \eta'_k}}{\Gamma(\sigma'_k)} - \frac{(t-s)^{\sigma_k - \eta_k}}{\Gamma(\sigma_k)} \right) w + \\ &\quad \sum_{j=1}^{k-1} p_{k-j}(t) \left(\frac{(t-s)^{\sigma'_k - \sigma'_j - \eta'_k}}{\Gamma(\sigma'_k)} - \frac{(t-s)^{\sigma_k - \sigma_j - \eta_k}}{\Gamma(\sigma_k)} \right) w. \end{aligned}$$

To prove the Lipschitz property, we use the following inequality

$$|\tau^{\delta'} - \tau^{\delta}| = \left| \int_{\delta}^{\delta'} \tau^s \ln \tau ds \right| \leq C(\delta_0, T) |\delta' - \delta|, \quad \forall \tau \in [0, T], \delta' \geq \delta \geq \delta_0 > 0.$$

Calculating directly gives

$$|K(t, s, z', w) - K(t, s, z, w)| \leq C(\eta_0, T) |z' - z|.$$

Now, we verify the Lipschitz condition of g_z where

$$g_z(t) = f(t) - p_k(t) \sum_{j=1}^k \frac{b_j t^{\sigma_j-1}}{\Gamma(\sigma_j)} - \sum_{j=1}^{k-1} p_{k-j}(t) \sum_{\ell=j+1}^k \frac{b_{\ell} t^{\sigma_{\ell}-\sigma_j-1}}{\Gamma(\sigma_{\ell}-\sigma_j)}.$$

Now, we consider the case $f \in C_{\gamma}(T; \mathbb{X})$. Multiplying t^{γ} to the last equality gives

$$t^{\gamma} g_z(t) = t^{\gamma} f(t) - p_k(t) \sum_{j=1}^k \frac{b_j t^{\gamma+\sigma_j-1}}{\Gamma(\sigma_j)} - \sum_{j=1}^{k-1} p_{k-j}(t) \sum_{\ell=j+1}^k \frac{b_{\ell} t^{\gamma+\sigma_{\ell}-\sigma_j-1}}{\Gamma(\sigma_{\ell}-\sigma_j)}.$$

We have $\gamma + \sigma_j - 1 \geq \gamma + \eta_0 - 1 > 0$, $\gamma + \sigma_{\ell} - \sigma_j - 1 \geq \gamma + \eta_0 - 1 > 0$ for every $j = 1, \dots, k$. Hence, using the same estimate as for $|K(t, s, z', w) - K(t, s, z, w)|$ we obtain

$$\|g_{z'} - g_z\|_{C_{\gamma}(T; \mathbb{X})} \leq C(\eta_0, T) \sum_{j=1}^k |b_j| |z' - z|.$$

From the above estimates we can apply Theorem 3.2, Part (c) to get the result in $C_{\gamma}(T; \mathbb{X})$. The L^p -case is similar. Hence, we omit it. \square

3.3 Abel linear equations of the first kind

In this subsection, we apply the order-continuity results obtained in the previous subsections to the Abel linear equations of first kind. We recall that $\Delta_T = \{(t, s, \alpha, z) : 0 \leq s \leq t \leq T, \alpha \in [\alpha_0, \alpha_1], z \in P\}$. For $K_0 : \Delta_T \rightarrow \mathbb{R}$, we define

$$\mathcal{A}_{\alpha, z} v(x) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{K_0(t, s, \alpha, z) v(s)}{(t-s)^{1-\alpha}} ds.$$

In this subsection, we shall investigate the continuity-with respect to the parameters α, z -of solutions of linear Abel equations which has the form

$$\mathcal{A}_{\alpha, z} v(x) = f(t), \quad 0 \leq t \leq T. \quad (3.15)$$

Theorem 3.4. *Let $\gamma \in [0, 1)$, $0 < \alpha_0 < \alpha_1 < 1$, $\alpha \in [\alpha_0, \alpha_1]$, $z \in P \subset \mathbb{R}^k$, $\mu, \nu \in (0, 1]$, $\kappa_1 > 0$, assume the following*

(i) $K_0 \in C(\Delta_T)$, $K_0(t, t, \alpha, z) = 1$;

(ii) $\left\| \frac{\partial K_0}{\partial t} \right\|_{L^{\infty}(\Delta_T)} \leq M$;

$$(iii) \sup_{0 \leq s \leq t \leq 1} \left| \frac{\partial K_0}{\partial t}(t, s, \alpha', z') - \frac{\partial K_0}{\partial t}(t, s, \alpha, z) \right| \leq \kappa_1(|\alpha' - \alpha|^\nu + |z' - z|^\mu);$$

and denote

$$\begin{aligned} H(t, s, \alpha, z) &= K_0(t, s, \alpha, z) - K_0(s, s, \alpha, z), \quad (t, s, \alpha, z) \in \Delta_T, \\ L_{\alpha, z}(t, s) &= -\frac{\sin \pi \alpha}{\pi} \int_s^t (t - \tau)^{-\alpha} (\tau - s)^{\alpha-1} ((\alpha - 1)(\tau - s)^{-1} H(\tau, s, \alpha, z) + \frac{\partial H}{\partial t}(\tau, s, \alpha, z)) d\tau, \\ B_{\alpha, z} u_{\alpha, z}(t) &= \int_0^t u_{\alpha, z}(s) L_{\alpha, z}(t, s) ds. \end{aligned}$$

(a) Let $f \in L^1(0, T; \mathbb{X})$ satisfy $D^\alpha f \in L^p(0, T; \mathbb{X})$. Then the equation (3.15) has a unique solution $u_{\alpha, z} \in L^p(0, T; \mathbb{X})$ such that

$$u_{\alpha, z}(t) = D_t^\alpha f + B_{\alpha, z} u_{\alpha, z}.$$

This solution satisfies

$$\|u_{\alpha, z}\|_{L^p(0, T; \mathbb{X})} \leq C(MT, p) \|D_t^\alpha f\|_{L^p(0, T; \mathbb{X})}$$

$$\text{where } M = \left\| \frac{\partial K_0}{\partial t} \right\|_{L^\infty(\Delta_T)}.$$

(b) If there is an $\alpha_2 \in (\alpha_1, 1)$ such that $D_t^{\alpha_2} f \in L^p(0, T; \mathbb{X})$ then there a constant $C = C(\alpha_0, \alpha_1, P)$ such that

$$\|u_{\alpha', z'} - u_{\alpha, z}\|_{L^p(0, T; \mathbb{X})} \leq C(|\alpha' - \alpha|^\nu + |z' - z|^\mu).$$

In addition, if (α_n, z_n) are random as in Theorem 3.2 then

$$\mathbb{E} \|u_{\alpha_n, z_n} - u_{\alpha, z}\|_{L^p(0, T; \mathbb{X})} \leq C \left((\mathbb{E} |\alpha_n - \alpha|^\lambda)^{\nu/\lambda} + \sum_{j=1}^k (\mathbb{E} |z_{jn} - \beta_j|^\rho)^{\mu_j/\rho} \right).$$

Proof. Proof of (a): From [16, page 86,87], we have the equality

$$\mathcal{A}_{\alpha, z} = J^\alpha (I - B_{\alpha, z}).$$

Hence, we obtain

$$u_{\alpha, z}(t) = D^\alpha f + B_{\alpha, z} u_{\alpha, z}.$$

Applying Theorem 3.1 we obtain the desired result.

Proof of (b): We first estimate $\|D^{\alpha'} f(t) - D^\alpha f(t)\|$. Assume that $\alpha_0 \leq \alpha \leq \alpha' \leq \alpha_1$. Since $\alpha_2 - \alpha', \alpha_2 - \alpha \geq \alpha_2 - \alpha_1$, we have in view of Lemma 3.1 (b)

$$\begin{aligned} \|D^{\alpha'} f - D^\alpha f\|_{L^p(0, T; \mathbb{X})} &= \left\| J^{\alpha_2 - \alpha'} D^{\alpha_2} f - J^{\alpha_2 - \alpha} D^{\alpha_2} f \right\|_{L^p(0, T; \mathbb{X})} \\ &\leq C(\alpha_2 - \alpha_1) \|D^{\alpha_2} f\|_{L^p(0, T; \mathbb{X})} |\alpha' - \alpha|. \end{aligned}$$

Put $\tau = s + \theta(t - s)$, we obtain

$$L_{\alpha, z}(t, s) = -\frac{(t - s)^{2-2\alpha} \sin \pi \alpha}{\pi} \int_0^1 (1 - \theta)^{-\alpha} \theta^{1-\alpha} g(t, s, \alpha, z, \theta) d\theta$$

where

$$g(t, s, \alpha, z, \theta) = (\alpha - 1)\theta^{-1}(t - s)^{-1}H(s + \theta(t - s), s, \alpha, z) + \frac{\partial H}{\partial \tau}(s + \theta(t - s), s, \alpha, z).$$

Since

$$|H(t, s, \alpha, z)| = |K_0(t, s, \alpha, z) - K_0(s, s, \alpha, z)| \leq M|s - t|,$$

the Lagrange theorem implies that

$$|g(t, s, \alpha, z, \theta)| \leq 2M.$$

We also claim that

$$|L_{\alpha', z'}(t, s) - L_{\alpha, z}(t, s)| \leq C(|\alpha' - \alpha|^\nu + |z' - z|^\mu). \quad (3.16)$$

In fact, the assumption (iii) implies

$$\begin{aligned} |H(s + \theta(t - s), s, \alpha', z') - H(s + \theta(t - s), s, \alpha, z)| &\leq 2\kappa_1 M(|\alpha' - \alpha|^\nu + |z' - z|^\mu)\theta|t - s|, \\ \left| \frac{\partial H}{\partial \tau}(s + \theta(t - s), s, \alpha', z') - \frac{\partial H}{\partial \tau}(s + \theta(t - s), s, \alpha, z) \right| &\leq 2\kappa_1(|\alpha' - \alpha|^\nu + |z' - z|^\mu). \end{aligned}$$

Hence,

$$|g(t, s, \alpha', z', \theta) - g(t, s, \alpha, z, \theta)| \leq 2\kappa_1(1 + M|\alpha - 1|)(|\alpha' - \alpha|^\nu + |z' - z|^\mu).$$

Substituting into the formula of $L_{\alpha', z'}(t, s) - L_{\alpha, z}(t, s)$ gives

$$\begin{aligned} |L_{\alpha', z'}(t, s) - L_{\alpha, z}(t, s)| &\leq \frac{2M}{\pi} \int_0^1 |\sin \pi \alpha' (1 - \theta)^{-\alpha'} \theta^{1-\alpha'} - \sin \pi \alpha (1 - \theta)^\alpha \theta^{1-\alpha}| d\theta + \\ &\quad \frac{2\kappa_1(1 + M|\alpha - 1|)}{\pi} (|\alpha' - \alpha|^\nu + |z' - z|^\mu) \int_0^1 \sin \pi \alpha (1 - \theta)^\alpha \theta^{1-\alpha} d\theta \end{aligned}$$

which gives directly the inequality (3.16). From the estimates for $D^{\alpha'} f - D^\alpha f$ and $L_{\alpha, z}$, we can use Part (d) of Theorem 3.2 to obtain the final inequality of Theorem. \square

3.4 A special Abel integral equation

In this section we use the notation $\phi_\lambda(t, \alpha)$ for the function which is a solution to the equation

$$\partial_t^\alpha \phi_\lambda = \lambda \phi_\lambda, \quad \phi_\lambda(0) = 1.$$

We can transform this equation into the Abel integral equation (see, e.g., [16, page 131])

$$\phi_\lambda(t, \alpha) = 1 + \lambda J^\alpha \phi_\lambda(t, \alpha).$$

Hence the function ϕ_λ can be represented using the function $E_{\alpha, 1}(z)$. For $\lambda > 0$, $\alpha > 0$, and $t > 0$ we have

$$\phi_\lambda(t, \alpha) = \phi_\lambda(t) = E_{\alpha, 1}(\lambda t^\alpha).$$

Using Theorem 3.2 with

$$K(t, s, \alpha, w) = \frac{\lambda}{\Gamma(\alpha)}(t - s)^{\alpha-1}w$$

and $\kappa = \frac{|\lambda|}{\Gamma(\alpha_0)}$ we can get the Lipschitz continuity –with respect to $\alpha \in [\alpha_0, \alpha_1]$ – of the function $\phi_\lambda(t, \alpha)$ as follows

$$\|\phi_\lambda(\cdot, \alpha') - \phi_\lambda(\cdot, \alpha)\|_{C[0, T]} \leq C|\alpha' - \alpha|E_{\alpha_0, 1}^2(|\lambda|\Gamma(\alpha_1)\max\{T^{\alpha_0}, T^{\alpha_1}\}/\Gamma(\alpha_0)). \quad (3.17)$$

Lemma 2.3 shows that the Lipschitz constant of the inequality is of order $e^{C'T|\lambda|^{1/\alpha}}$ which is very large when $\lambda \rightarrow \infty$. Hence, we will look for a better estimate for the case $\lambda < 0$. In fact we have

Lemma 3.3. *Letting $\lambda < 0$, we obtain the following estimates*

(a) *For $0 < \alpha \leq 1$, we have*

$$|\phi_\lambda(t_1, \alpha) - \phi_\lambda(t_2, \alpha)| \leq C|\lambda| \cdot |t_1 - t_2|^\alpha.$$

(b) *For $0 < \alpha_0 < \alpha_1 < 1, \lambda < -1$ and $\alpha, \alpha' \in [\alpha_0, \alpha_1], \alpha < \alpha'$ and let $\beta, \beta' \in [\beta_0, \beta_1]$. Then there exist constants $C = C(\alpha_0, \alpha_1, \beta_0, \beta_1)$ such that*

$$\begin{aligned} |\phi_\lambda(t, \alpha') - \phi_\lambda(t, \alpha)| &\leq C|\lambda| \cdot |\alpha' - \alpha|, \\ |E_{\alpha', 1}(-|\lambda|^{\beta'} t^{\alpha'}) - E_{\alpha, 1}(-|\lambda|^\beta t^\alpha)| &\leq C|\lambda|^{\beta_1} |\ln \lambda| (|\beta' - \beta| + |\alpha' - \alpha|) \end{aligned}$$

for $t \in [0, T]$.

(c) *Letting $f \in L^2(0, T)$, we put*

$$G_{\alpha, \lambda}(t) = \int_0^t (t-s)^{\alpha-1} E_{\alpha, \alpha}(\lambda(t-s)^\alpha) f(s) ds.$$

Then, we have the following estimate

$$\|G_{\alpha', \lambda'} - G_{\alpha, \lambda}(t)\|_{L^2(0, T)} \leq C \left[|\alpha' - \alpha|(1 + |\lambda|) + |\lambda' - \lambda| \right] \|f(t)\|_{L^2(0, T)}. \quad (3.18)$$

Proof. Proof of (a): From Lemma 2.3 (a) the inequality $|t_1^\alpha - t_2^\alpha| \leq |t_2 - t_1|^\alpha$ ($0 \leq \alpha \leq 1$), we have

$$\begin{aligned} |\phi_\lambda(t_1, \alpha) - \phi_\lambda(t_2, \alpha)| &= |E_{\alpha, 1}(\lambda t_1^\alpha) - E_{\alpha, 1}(\lambda t_2^\alpha)| \\ &\leq C|\lambda| |t_1^\alpha - t_2^\alpha| \leq C|\lambda| |t_1 - t_2|^\alpha. \end{aligned}$$

Proof of (b): Using Lemma 2.3 (a) gives

$$\begin{aligned} |\phi_\lambda(t, \alpha') - \phi_\lambda(t, \alpha)| &\leq C|\lambda| \cdot |t^{\alpha'} - t^\alpha| \\ &\leq C|\lambda| \sup_{\alpha_0 \leq \alpha \leq \alpha_1} t^\alpha |\ln t| |\alpha' - \alpha| \\ &\leq C'|\lambda| \cdot |\alpha' - \alpha|. \end{aligned}$$

Finally, we have

$$\begin{aligned} |E_{\alpha', 1}(-|\lambda|^{\beta'} t^{\alpha'}) - E_{\alpha, 1}(-|\lambda|^\beta t^\alpha)| &\leq |E_{\alpha', 1}(-|\lambda|^{\beta'} t^{\alpha'}) - E_{\alpha', 1}(-|\lambda|^\beta t^{\alpha'})| \\ &\quad + |E_{\alpha', 1}(-|\lambda|^\beta t^{\alpha'}) - E_{\alpha, 1}(-|\lambda|^\beta t^\alpha)| \\ &\leq C \left| |\lambda|^{\beta'} t^{\alpha'} - |\lambda|^\beta t^{\alpha'} \right| + C|\lambda|^\beta |\alpha' - \alpha| \\ &\leq CT^{\alpha_1} \left| |\lambda|^{\beta'} - |\lambda|^\beta \right| + C|\lambda|^\beta |\alpha' - \alpha| \\ &\leq CT^{\alpha_1} |\lambda|^{\beta'} |\ln(|\lambda|)|\beta' - \beta| + C\lambda^\beta |\alpha' - \alpha|. \end{aligned}$$

Proof of (c): For $\lambda', \lambda < 0$, we have

$$\begin{aligned}
G_{\alpha', \lambda'}(t) - G_{\alpha, \lambda}(t) &= \int_0^t [(t-s)^{\alpha'-1} - (t-s)^{\alpha-1}] E_{\alpha', \alpha'}(\lambda'(t-s)^{\alpha'}) f(s) ds \\
&+ \int_0^t (t-s)^{\alpha-1} [E_{\alpha', \alpha'}(\lambda'(t-s)^{\alpha'}) - E_{\alpha', \alpha'}(\lambda(t-s)^{\alpha'})] f(s) ds \\
&+ \int_0^t (t-s)^{\alpha-1} [E_{\alpha', \alpha'}(\lambda(t-s)^{\alpha'}) - E_{\alpha, \alpha}(\lambda(t-s)^{\alpha'})] f(s) ds \\
&+ \int_0^t (t-s)^{\alpha-1} [E_{\alpha, \alpha}(\lambda(t-s)^{\alpha'}) - E_{\alpha, \alpha}(\lambda(t-s)^{\alpha})] f(s) ds.
\end{aligned}$$

Inserting appropriate terms deduces

$$\|G_{\alpha', \lambda'} - G_{\alpha, \lambda}\|_{L^2(0, T)}^2 \leq 4(K_1^2 + K_2^2 + K_3^2 + K_4^2) \|f\|_{L^2(0, T)}^2$$

where

$$\begin{aligned}
K_1 &= \int_0^T |s^{\alpha'-1} - s^{\alpha-1}| E_{\alpha', \alpha'}(\lambda' s^{\alpha'}) ds, \\
K_2 &= \int_0^T |s^{\alpha-1} [E_{\alpha', \alpha'}(\lambda' s^{\alpha'}) - E_{\alpha', \alpha'}(\lambda s^{\alpha'})]| ds, \\
K_3 &= \int_0^T |s^{\alpha-1} [E_{\alpha', \alpha'}(\lambda s^{\alpha'}) - E_{\alpha, \alpha}(\lambda s^{\alpha'})]| ds, \\
K_4 &= \int_0^T |s^{\alpha-1} [E_{\alpha, \alpha}(\lambda s^{\alpha'}) - E_{\alpha, \alpha}(\lambda s^{\alpha})]| ds.
\end{aligned}$$

Using Lemma 3.1 (a) gives

$$K_1 \leq C \int_0^T |s^{\alpha'-1} - s^{\alpha-1}| ds \leq C |\alpha' - \alpha|.$$

For K_2 , we use Lemma 2.3 (a) to obtain

$$\begin{aligned}
K_2 &\leq \int_0^T |s^{\alpha-1} [E_{\alpha', \alpha'}(\lambda' s^{\alpha'}) - E_{\alpha', \alpha'}(\lambda s^{\alpha'})]| ds \\
&\leq C |\lambda' - \lambda| \int_0^T s^{\alpha-1} s^{\alpha'} ds \leq C' |\lambda' - \lambda|.
\end{aligned}$$

Similarly

$$K_4 \leq C |\lambda| \cdot |\alpha' - \alpha|.$$

Finally, for $z \leq 0$

$$|E_{\alpha', \alpha'}(z) - E_{\alpha, \alpha}(z)| \leq |E_{\alpha', \alpha'}(z) - E_{\alpha, \alpha'}(z)| + |E_{\alpha, \alpha'}(z) - E_{\alpha, \alpha}(z)| \leq C |\alpha' - \alpha|$$

where

$$C = \sup \left\{ \left| \frac{\partial E_{\alpha, \beta}}{\partial \alpha}(z) \right| + \left| \frac{\partial E_{\alpha, \beta}}{\partial \beta}(z) \right| : (\alpha, \beta, z) \in [\alpha_0, \alpha_1] \times [\beta_0, \beta_1] \times (-\infty, 0] \right\}.$$

Hence, we can use Lemma 2.3 (a) to obtain

$$K_3 \leq \int_0^T \left| s^{\alpha-1} [E_{\alpha',\alpha'}(\lambda s^{\alpha'}) - E_{\alpha,\alpha}(\lambda s^{\alpha})] \right| ds \leq C|\alpha' - \alpha|.$$

Combining the estimates for K_1, K_2, K_3, K_4 gives

$$\|G_{\alpha',\lambda'} - G_{\alpha,\lambda}\|_{L^2(0,T)} \leq C \left[|\alpha' - \alpha|(1 + |\lambda|) + |\lambda' - \lambda| \right] \|f\|_{L^2(0,T)}.$$

This completes the proof of (c). \square

4 Continuity of the solutions of some space-time fractional partial differential equations.

In this section, we will consider the continuity of solutions of some abstract partial differential equations with respect to the parameters β, α of some fractional partial differential equations.

4.1 The abstract fractional diffusion equation in a Banach space

We first investigate the continuity properties in a Banach space. To this end, an outline for classical definitions in the theory of semigroup on Banach spaces is necessary. Let \mathbb{X} be a Banach space as the in previous section and $\mathcal{L}(\mathbb{X})$ be the set of bounded linear operators on \mathbb{X} . Let $B : D(B) \rightarrow \mathbb{X}$ ($D(B) \subset \mathbb{X}$) be a closed linear operator. We denote (see [13], page 55) the spectrum set of B , the resolvent set of B by

$$\begin{aligned} \sigma(B) &= \left\{ \lambda \in \mathbb{C} : \lambda - B \text{ is not bijective} \right\}, \\ \rho(B) &= \left\{ \lambda \in \mathbb{C} : \lambda \notin \sigma(B) \right\} \end{aligned}$$

respectively. For $0 \leq \omega < \pi$ we denote the sector with angle ω by

$$\Sigma_\omega = \{ \lambda \in \mathbb{C} : \lambda \neq 0, |\arg \lambda| < \omega \}.$$

As in [33, page 91] we say that the operator B is positive if $[0, \infty) \subset \rho(-B)$ and

$$\sup_{\lambda \geq 0} \|(\lambda + 1)(\lambda I + B)^{-1}\|_{\mathcal{L}(\mathbb{X})} < \infty.$$

We denote by Φ_B the set of real numbers $\eta \in (0, \pi]$ satisfying $\overline{\Sigma_\eta} \subset \rho(-B)$ and

$$\sup_{\lambda \in \overline{\Sigma_\eta}} \|(\lambda + 1)(\lambda I + B)^{-1}\|_{\mathcal{L}(\mathbb{X})} < \infty.$$

From the positivity of B , we can verify that $\Phi_B \neq \emptyset$. We define the spectral angle of B by

$$\phi_B = \inf \{ \omega \in (0, \pi] : \pi - \omega \in \Phi_B \}.$$

As in [6] we consider two Banach spaces $X_1 \subset X_0$ such that X_0 is dense in X_1 and that $A : X_1 \rightarrow X_0$ is an isomorphism. We denote by $X_\theta = [X_0, X_1]_\theta$ ($0 \leq \theta \leq 1$) the interpolation spaces between X_0 and X_1 defined by the following: element $\xi \in X_\theta$ if and only if

$$\lim_{|\lambda| \rightarrow \infty, |\arg \lambda| < \eta} \|\lambda^\theta A(\lambda I + A)^{-1} \xi\|_{X_0} = 0.$$

for every $0 \leq \eta < \pi - \phi_A$. With η fixed, The space X_θ , called the abstract Hölder space (see [13], page 130), is a Banach space with the norm

$$\|\xi\|_{X_\theta} = \sup_{|\arg \lambda| < \eta, \lambda \neq 0} \|\lambda^\theta A(\lambda I + A)^{-1} \xi\|_{X_0}.$$

We can verify directly that $X_{\theta_1} \cap X_{\theta_2} \subset X_\theta$ for $0 \leq \theta_2 \leq \theta \leq \theta_1 \leq 1$.

For $\xi \in X_0$, we consider the problem of finding $u : [0, T] \rightarrow X_1$ such that

$$D_t^\alpha(u - \xi) + Au = 0, \quad u(0) = \xi.$$

Let $\alpha \in (0, 2)$, $\gamma \in (0, 1)$. We define the following space of functions

$$\begin{aligned} C_{\gamma,0}(T, X) &= \{h : (0, T] \rightarrow \mathbb{X} : h \in C_\gamma(T, X), \lim_{t \rightarrow 0+} t^\gamma \|h(t)\| = 0\}, \\ C_{\gamma,0}^\alpha(T, X_0, X_1) &= \{h : h \in C_{\gamma,0}(T, X_1); D_t^\alpha h \in C_{\gamma,0}(T, X_0)\} \end{aligned}$$

with the respective norms

$$\begin{aligned} \|u\|_{C_{\gamma,0}(T, X)} &= \sup_{t \in (0, 1]} t^\gamma \|u(t)\|_X, \\ \|v\|_{C_{\gamma,0}^\alpha(T, X_0, X_1)} &= \sup_{t \in (0, 1]} t^\gamma (\|D_t^\alpha v(t)\|_{X_0} + \|v(t)\|_{X_1}). \end{aligned}$$

We define the trace operator

$$Tr : C_{\gamma,0}^\alpha(T, X_0, X_1) \rightarrow X_0 \quad \text{by } Tr(u) = u(0).$$

The next theorem establishes the existence, uniqueness and continuity of solution to time fractional diffusion type equations in a Banach space.

Theorem 4.1. (a) *Let $\alpha \in (0, 2)$, $\gamma \in (0, \min\{1, \alpha\})$. We assume that A is nonnegative with the spectral angle ϕ_A satisfying*

$$0 < \phi_A < \pi \left(1 - \frac{\alpha}{2}\right).$$

Put $\hat{\gamma} = \frac{\gamma}{\alpha}$. If $\xi \in X_{1-\hat{\gamma}}$ then

$$Tr(C_{\gamma,0}^\alpha(T, X_0, X_1)) = X_{1-\hat{\gamma}}$$

and the problem

$$D_t^\alpha(u - \xi) + Au = 0$$

has a unique solution $u_\alpha \in C_{\gamma,0}^\alpha(T, X_0, X_1)$ given by

$$u_\alpha(t) = \frac{1}{2\pi i} \int_{\gamma_{1,\varphi}} e^{\lambda t} (\lambda^\alpha I + A)^{-1} \lambda^{\alpha-1} \xi d\lambda.$$

Here $\varphi \in \left(\frac{\pi}{2}, \frac{\pi - \phi_A}{\alpha}\right)$ and the curve $\gamma_{1,\varphi}$ is defined in (2.2).

(b) *For $0 < \alpha_0 < \alpha_1 < 2$, $\alpha \in [\alpha_0, \alpha_1]$, $\gamma \in (0, 1)$ and $0 < \gamma < \min\{1, \alpha_0\}$, if A is nonnegative with the spectral angle ϕ_A satisfying*

$$0 < \phi_A < \pi \left(1 - \frac{\alpha_1}{2}\right),$$

then

$$\text{Tr}(C_{\gamma,0}^\alpha(T, X_0, X_1)) = X_{1-\hat{\gamma}}$$

for every $0 < \gamma < \min\{1, \alpha_0\}$. Moreover, put

$$\hat{\gamma}_0 = \frac{\gamma}{\alpha_0}, \hat{\gamma}_1 = \frac{\gamma}{\alpha_1}.$$

If $\xi \in X_{1-\hat{\gamma}_0} \cap X_{1-\hat{\gamma}_1}$ and $\varphi \in \left(\frac{\pi}{2}, \frac{\pi-\phi_A}{\alpha_1}\right)$ then the problem

$$D_t^\alpha(u - \xi) + Au = 0$$

has a unique solution $u_{\alpha,\xi} \in C_{\gamma,0}^\alpha(T, X_0, X_1)$ satisfying

$$u_{\alpha,\xi}(t) = \frac{1}{2\pi i} \int_{\gamma_{1,\varphi}} e^{\lambda t} (\lambda^\alpha I + A)^{-1} \lambda^{\alpha-1} \xi d\lambda$$

for every $\alpha \in [\alpha_0, \alpha_1]$, $\varphi \in \left(\frac{\pi}{2}, \frac{\pi-\phi_A}{\alpha_1}\right)$. Moreover, there exists a constant $C = C(\alpha_0, \alpha_1)$ such that

$$\|u_{\alpha',\xi'}(t) - u_{\alpha,\xi}(t)\|_{X_0} \leq C(1 + \|\xi\|_{X_0}) (|\alpha' - \alpha| + \|\xi' - \xi\|_{X_0})$$

for all $\alpha, \alpha' \in [\alpha_0, \alpha_1]$, $\xi, \xi' \in X_{1-\hat{\gamma}_0} \cap X_{1-\hat{\gamma}_1}$.

Proof. Proof of (a): Readers can be found in Clement et al. [5].

Proof of (b): Since A is nonnegative with the spectral angle ϕ_A satisfying $0 < \phi_A < \pi(1 - \frac{\alpha_1}{2})$ and $\alpha \in [\alpha_0, \alpha_1]$ we have $0 < \phi_A < \pi(1 - \frac{\alpha}{2})$. Hence we obtain

$$\text{Tr}(C_{\gamma,0}^\alpha(T, X_0, X_1)) = X_{1-\hat{\gamma}}$$

as in (a). Since $X_{1-\hat{\gamma}_0} \cap X_{1-\hat{\gamma}_1} \subset X_{1-\hat{\gamma}}$, if $\xi \in X_{1-\hat{\gamma}_0} \cap X_{1-\hat{\gamma}_1}$ then $\xi \in X_{1-\hat{\gamma}}$. Applying (a), we have

$$u_\alpha(t) = \frac{1}{2\pi i} \int_{\gamma_{1,\varphi}} e^{\lambda t} (\lambda^\alpha I + A)^{-1} \lambda^{\alpha-1} \xi d\lambda \quad \text{for every } \alpha \in [\alpha_0, \alpha_1].$$

For $\lambda \in \gamma_{1,\varphi}$, $|\lambda| > 1$, we have $\arg(\lambda) = \pm\varphi$. It follows that

$$|\arg \lambda^\alpha| = \varphi\alpha \leq \varphi\alpha_1 < \pi - \phi_A \quad \text{for } \lambda \in \gamma_{1,\varphi}, |\lambda| > 1, \alpha \in [\alpha_0, \alpha_1].$$

Hence

$$\begin{aligned} \sup_{\lambda \in \gamma_{1,\varphi}} \|(\lambda^\alpha I + A)^{-1}\| &\leq \sup_{\lambda \in \gamma_{1,\varphi}} \|\lambda^\alpha (\lambda^\alpha I + A)^{-1}\| \\ &\leq C_{\omega_0} \sup_{\lambda \in \gamma_{1,\varphi}} \|(1 + \lambda^\alpha)(\lambda^\alpha I + A)^{-1}\| \leq C_{\omega_0} M_{\omega_0} \end{aligned} \quad (4.1)$$

where $\omega_0 = \pi - \varphi\alpha_1$, $C_{\omega_0} = \sup_{\lambda \in \overline{\Sigma_{\omega_0}}} |\lambda(\lambda + 1)^{-1}|$ and

$$M_{\omega_0} = \sup_{\lambda \in \overline{\Sigma_{\omega_0}}} \|(\lambda + 1)(\lambda I + A)^{-1}\|.$$

We consider the last inequality. For $\alpha, \alpha' \in [\alpha_0, \alpha_1]$ we can write

$$u_{\alpha',\xi'}(t) - u_{\alpha,\xi}(t) = J_1 + J_2 + J_3$$

where

$$\begin{aligned} J_1 &= \frac{1}{2\pi i} \int_{\gamma_{1,\varphi}} e^{\lambda t} ((\lambda^{\alpha'} I + A)^{-1} - (\lambda^\alpha I + A)^{-1}) \lambda^{\alpha'-1} \xi' d\lambda, \\ J_2 &= \frac{1}{2\pi i} \int_{\gamma_{1,\varphi}} e^{\lambda t} (\lambda^\alpha I + A)^{-1} (\lambda^{\alpha'-1} - \lambda^{\alpha-1}) \xi' d\lambda, \\ J_3 &= \frac{1}{2\pi i} \int_{\gamma_{1,\varphi}} e^{\lambda t} (\lambda^\alpha I + A)^{-1} \lambda^{\alpha-1} (\xi' - \xi) d\lambda. \end{aligned}$$

From (4.1) we have

$$\|J_3\| \leq \frac{1}{2\pi} \int_{\gamma_{1,\varphi}} e^{Re\lambda t} C_{\omega_0} M_{\omega_0} \|\xi' - \xi\|_{X_0} |d\lambda| \leq C \|\xi' - \xi\|_{X_0}.$$

Using $(\lambda^\alpha I + A)^{-1} - (\lambda^{\alpha'} I + A)^{-1} = (\lambda^{\alpha'} - \lambda^\alpha)(\lambda^\alpha I + A)^{-1}(\lambda^{\alpha'} I + A)^{-1}$ and estimating directly J_1 , we obtain in view of (4.1)

$$\|J_1\|_{X_0} \leq \frac{1}{2\pi} \int_{\gamma_{1,\varphi}} e^{Re\lambda t} |\lambda^{\alpha'} - \lambda^\alpha| C_{\omega_0}^2 M_{\omega_0}^2 \|\xi'\|_{X_0} |d\lambda| \leq C |\alpha' - \alpha| \|\xi'\|_{X_0}.$$

Similarly,

$$\|J_2\|_{X_0} \leq \frac{1}{2\pi} \int_{\gamma_{1,\varphi}} e^{Re\lambda t} |\lambda^{\alpha'} - \lambda^\alpha| C_{\omega_0} M_{\omega_0} \|\xi'\|_{X_0} |d\lambda| \leq C |\alpha' - \alpha| \|\xi'\|_{X_0}.$$

This completes the proof of the theorem. \square

4.2 The abstract fractional diffusion equation in a Hilbert space

In this section we will study the existence, uniqueness and the continuity of solutions to time fractional equation in a Hilbert space. The main results are theorems 4.2, 4.3, and 4.4.

We denote by V and H the real separable Hilbert spaces, V' the dual of V and $\langle \cdot, \cdot \rangle$ the inner product of H . We assume that the space V is dense in H and continuously embedded into H . Usually, we write $V \hookrightarrow H \hookrightarrow V'$ and

$$\varphi(v) := \langle \varphi, v \rangle_{V' \times V}, \quad \forall \varphi \in V', v \in V.$$

For $x \in H$, $\alpha \in (0, 1]$ we put

$$W_\alpha(x, V, H) := \{u \in L^2(0, T; V) : D_t^\alpha(u - x) \in L^2(0, T; V'), u(0) = x\}.$$

We note that $W_\alpha(x, V, H) \subset W_{\alpha'}(x, V, H)$ for $0 < \alpha' < \alpha \leq 1$. For $\gamma \in (0, 1)$, we denote by

$$H^\gamma(\mathbb{R}; V, V') = \{v : v \in L^2(\mathbb{R}; V), |\tau|^\gamma \hat{v} \in L^2(\mathbb{R}; V')\}$$

where $\hat{v}(\tau) = \int_{-\infty}^{\infty} v(t) e^{-it\tau} dt$. The space $H^\gamma(\mathbb{R}; V, V')$ is a Hilbert space with the norm $\|v\|_{H^\gamma(\mathbb{R}; V, V')}^2 = \|v\|_{L^2(\mathbb{R}; V)}^2 + \|\tau|^\gamma \hat{v}\|_{L^2(\mathbb{R}; V')}^2$. From the latter space we can define the space of all functions having the fractional derivatives of order γ by putting the set

$$H^\gamma(0, T; V, V') = \{w : w = v|_{(0, T)}, v \in H^\gamma(\mathbb{R}; V, V')\}.$$

We have, see e.g. [19], page 61,

$$H^\gamma(0, T; V, V') = \{w : w \in L^2(0, T; V), D_t^\gamma w \in L^2(0, T; V')\}.$$

We denote by K a compact subset in \mathbb{R}^k . For every $\beta \in K$, let $a_\beta(t, \cdot, \cdot) : V \times V \rightarrow \mathbb{R}$ be a bilinear functional and $f_\beta \in L^2(0, T; V')$. We define the operator $B_\beta(t) : V \rightarrow V'$ by

$$\langle B_\beta(t)v, w \rangle_{V' \times V} = a_\beta(t, w, v) \quad \forall v, w \in V.$$

We consider the problem of finding $u_{\alpha, \beta} \in W_\alpha(x, V, H)$ such that

$$\frac{d}{dt} \langle J^{1-\alpha}(u_{\alpha, \beta} - x), v \rangle + a_\beta(t, u_{\alpha, \beta}(t), v) = \langle f_\beta(t), v \rangle_{V' \times V}, \quad v \in V, a.a.t \in (0, T). \quad (4.2)$$

We have the theorem which establishes the existence, uniqueness and the continuity of solutions to time fractional diffusion type equation in a Hilbert space.

Theorem 4.2. (a) *Let T, M, c, d , be constants, $0 < \alpha_0 < \alpha_1 < 1$, $\alpha \in [\alpha_0, \alpha_1]$. We assume that $a_\beta(t, \cdot, \cdot) : V \times V \rightarrow \mathbb{R}$ is a bilinear functional satisfying A1–A2 below*

$$\mathbf{A1)} \quad |a_\beta(t, v, w)| \leq M \|v\|_V \|w\|_V, \quad \forall v, w \in V, \beta \in K,$$

$$\mathbf{A2)} \quad a_\beta(t, v, v) \geq c \|v\|_V^2 - d \|v\|_H^2, \quad \forall v \in V,$$

for a.a. $t \in (0, T)$.

Then, for $x \in H, f_\beta \in L^2(0, T; V')$ there exists a unique function $u_{\alpha, \beta} \in W_\alpha(x, V, H)$ satisfying (4.2).

Moreover, there exists a constant $M_0 > 0$ independent of α, β such that

$$\|D_t^\alpha(u_{\alpha, \beta} - x)\|_{L^2(0, T; V')} + \|u_{\alpha, \beta}\|_{L^2(0, T; V)} \leq M_0 (\|x\| + \|f_\beta\|_{L^2(0, T; V')}). \quad (4.3)$$

(b) *Let $\alpha_n \in [\alpha_0, \alpha_1], \beta_n \in K$ for every $n \in \mathbb{N}$ and*

$$\lim_{n \rightarrow \infty} \alpha_n = \alpha, \lim_{n \rightarrow \infty} \beta_n = \beta.$$

Assume A3 and A4, or A4' below hold

$$\mathbf{A3)} \quad \lim_{\beta_n \rightarrow \beta} \|f_{\beta_n} - f_\beta\|_{L^2(0, T; V')} = 0;$$

$$\mathbf{A4)} \quad \lim_{n \rightarrow \infty} \|B_{\beta_n}(\cdot)v - B_\beta(\cdot)v\|_{L^2(0, T; V')} = 0 \text{ for every } v \in V;$$

$$\mathbf{A4')} \quad \text{there is a subset } D \subset V \text{ such that } \text{span}\{D\} \text{ is dense in } H \text{ and that } \lim_{n \rightarrow \infty} \|B_{\beta_n}(\cdot)v - B_\beta(\cdot)v\|_{L^2(0, T; H')} = 0 \text{ for every } v \in D.$$

Then we have $u_{\alpha_n, \beta_n} \rightarrow u_{\alpha, \beta}$ as $n \rightarrow \infty$ in $L^2(0, T; V)$.

In addition, if V is compactly embedded in H then

$$\lim_{n \rightarrow \infty} \|u_{\alpha_n, \beta_n} - u_{\alpha, \beta}\|_{L^2(0, T; V')} = 0.$$

Proof. The proof of part (a) is given in [34]. We now prove part (b) of the theorem. By (4.3) there exist $u \in L^2(0, T; V)$ and a subsequence $u_{\alpha_{n_k}, \beta_{n_k}}$, still denote by u_{α_n, β_n} , such that

$$u_{\alpha_n, \beta_n} \rightarrow u \quad \text{as } n \rightarrow \infty \text{ in } L^2(0, T; V).$$

We will prove that u satisfies the equation

$$\langle D_t^\alpha(u - x), v \rangle + a_\beta(t, u, v) = \langle f_\beta, v \rangle, \quad \forall v \in V. \quad (4.4)$$

We have

$$-\int_0^T \varphi'(t) \langle J^{1-\alpha_n}(u_{\alpha_n, \beta_n} - x), v \rangle dt + \int_0^T \varphi(t) a_{\beta_n}(t, u_{\alpha_n, \beta_n}(t), v) dt = \int_0^T \varphi(t) \langle f_{\beta_n}(t), v \rangle_{V' \times V} dt.$$

We consider the first term of the equality

$$\begin{aligned} \int_0^T \varphi'(t) \langle J^{1-\alpha_n}(u_{\alpha_n, \beta_n} - x), v \rangle dt &= \int_0^T \varphi'(t) \langle (J^{1-\alpha_n} - J^{1-\alpha})(u_{\alpha_n, \beta_n} - x), v \rangle dt \\ &+ \int_0^T \varphi'(t) \langle J^{1-\alpha} u_{\alpha_n, \beta_n}, v \rangle dt \equiv I_{1n} + I_{2n}. \end{aligned}$$

We have

$$|I_{1,n}| \leq C \|\varphi'\|_{L^\infty(0,T)} \|v\|_{L^2(0,T;V)} \|(J^{1-\alpha_n} - J^{1-\alpha})(u_{\alpha_n, \beta_n} - x)\|_{L^2(0,T;V')}.$$

Using Lemma 3.1 (b) gives

$$|I_{1,n}| \leq C \|\varphi'\|_{L^\infty(0,T)} \|v\|_{L^2(0,T;V)} |\alpha_n - \alpha| \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

On the other hand, $I_{2n} \rightarrow \int_0^T \varphi'(t) \langle J^{1-\alpha} u, v \rangle dt$, Hence

$$\lim_{n \rightarrow \infty} \int_0^T \varphi'(t) \langle J^{1-\alpha_n} u_{\alpha_n, \beta_n}, v \rangle dt = \int_0^T \varphi'(t) \langle J^{1-\alpha} u, v \rangle dt.$$

Now, we consider the second term of (4.4). If A4) holds then

$$\begin{aligned} \int_0^T \varphi(t) a_{\beta_n}(t, u_{\alpha_n, \beta_n}, v) dt &= \int_0^T \langle u_{\alpha_n, \beta_n}, B_{\beta_n}(t)v - B_\beta(t)v \rangle_{V \times V'} dt + \int_0^T \varphi(t) \langle u_{\alpha_n, \beta_n}, B_\beta v \rangle_{V \times V'} dt \\ &\rightarrow \int_0^T \varphi(t) \langle u, B_\beta(t)v \rangle_{V \times V'} dt = \int_0^T \varphi(t) a_\beta(t, u, v) dt \quad \text{as } n \rightarrow \infty. \end{aligned}$$

If A4') holds then

$$\begin{aligned} \int_0^T \varphi(t) a_{\beta_n}(t, u_{\alpha_n, \beta_n}, v) dt &= \int_0^T \langle u_{\alpha_n, \beta_n}, B_{\beta_n}(t)v - B_\beta(t)v \rangle_{H \times H'} dt + \int_0^T \varphi(t) \langle u_{\alpha_n, \beta_n}, B_\beta v \rangle_{H \times H'} dt \\ &\rightarrow \int_0^T \varphi(t) \langle u, B_\beta(t)v \rangle_{H \times H'} dt = \int_0^T \varphi(t) a_\beta(t, u, v) dt \quad \text{as } n \rightarrow \infty. \end{aligned}$$

These limits imply that the function u is the weak solution of (4.4). By the uniqueness, we have $u = u_{\alpha, \beta}$. Since the limit holds for every subsequence of u_{α_n, β_n} , we obtain $u_{\alpha_n, \beta_n} \rightarrow u_{\alpha, \beta}$ as $n \rightarrow \infty$ in $L^2(0, T; V)$. Finally, for $\gamma \in (0, \alpha_0)$, we have $u_{\alpha_n, \beta_n}, u_{\alpha, \beta} \in H^\gamma(0, T; V, V')$. If V is compactly embedded into H then Theorem 5.2 in [19], page 61, implies that $H^\gamma(0, T; V, V')$ is compactly embedded in $L^2(0, T; V')$. Hence

$$\lim_{n \rightarrow \infty} \|u_{\alpha_n, \beta_n} - u_{\alpha, \beta}\|_{L^2(0,T;V')} = 0.$$

□

4.3 Continuity of time fractional equations with a countable spectrum in a Hilbert space

The continuity of the solutions in Theorem 4.2 are very weak. To get the stronger continuity, we consider an **initial value problem in a Hilbert space**. In particular we consider operators that have a countable spectrum. Let H be a Hilbert space with the inner-product $\langle \cdot, \cdot \rangle$ and the norm $\|\cdot\|$. Let $A : D(A) \rightarrow H$ be an operator defined on the subset $D(A)$ (called the domain of

A) which is dense in H . We assume that A has the eigenvectors $\phi_p \in D(A)$ corresponding to the eigenvalues λ_p , i.e.,

$$A\phi_p = \lambda_p\phi_p \text{ for } p = 1, 2, \dots$$

We also assume that $0 < \lambda_p \leq \lambda_{p+1}$, $\lim_{p \rightarrow \infty} \lambda_p = \infty$ and that $\{\phi_p\}$ is an orthonormal basis of H . For $s \geq 0$, we denote by $H^s \subset H$ the Hilbert space as collection of function with the bounded norm induced by the following

$$\|v\|_{H^s} = \sqrt{\sum_{p=1}^{\infty} \lambda_p^{2s} |\langle v, \phi_p \rangle|^2},$$

where $H^0 = H$ and $H^1 = D(A)$. For $\gamma \geq 0$, we denote by $L_{\gamma,s}^2(T)$ the Hilbert space of functions $f : (0, T) \rightarrow H^s$ such that

$$\|f\|_{L_{\gamma,s}^2(T)}^2 = \int_0^T t^{2\gamma} \|f(t)\|_{H^s}^2 dt < \infty.$$

For a constant $\beta > 0$, using the spectral theory, (fractional) powers A^β can be defined by

$$A^\beta v := \sum_{k=1}^{\infty} \lambda_k^\beta \langle v, \phi_k \rangle \phi_k, \quad (4.5)$$

with $D(A^\beta) = H^\beta$. For $\theta \in H^\beta$, $f \in L^2(0, T; H)$, we consider the forward problem for inhomogeneous time-fractional diffusion

$$\begin{cases} \partial_t^\alpha u_{\alpha,\beta,\theta,f} + A^\beta u_{\alpha,\beta,\theta,f} = f(t), & t \in (0, T), \\ u_{\alpha,\beta,\theta,f}(0) = \theta, \end{cases} \quad (4.6)$$

which can be rewritten as

$$D_t^\alpha(u_{\alpha,\beta,\theta,f} - \theta) + A^\beta u_{\alpha,\beta,\theta,f} = f(t), \quad t \in (0, T).$$

We can define the weak solution of the problem as in the general case with $V = H^{\beta/2}$ and

$$a_\beta(t; u, v) = \sum_{p=1}^{\infty} \lambda_p^\beta \langle u, \phi_p \rangle \langle v, \phi_p \rangle \quad \forall u, v \in V.$$

To consider the problem (4.6) conveniently, we write $u_{\alpha,\beta,\theta,f} = v_{\alpha,\beta,\theta} + w_{\alpha,\beta,f}$ where $v_{\alpha,\beta,\theta} : (0, T) \rightarrow H$ satisfies the homogeneous problem

$$\begin{cases} \partial_t^\alpha v_{\alpha,\beta,\theta} + A^\beta v_{\alpha,\beta,\theta} = 0, & t \in (0, T), \\ v_{\alpha,\beta,\theta}(0) = \theta, \end{cases} \quad (4.7)$$

and $w_{\alpha,\beta,f} : (0, T) \rightarrow H$ satisfies the nonhomogeneous problem with zero initial value

$$\begin{cases} \partial_t^\alpha w_{\alpha,\beta,f} + A^\beta w_{\alpha,\beta,f} = f(t), & t \in (0, T), \\ w_{\alpha,\beta,f}(0) = 0. \end{cases} \quad (4.8)$$

We have the following theorem which establishes the existence, uniqueness, and continuity of solutions of the homogeneous equation (4.7).

Theorem 4.3. Let $s \geq 0, r \geq 0, 0 < \alpha_0 < \alpha_1 < 1, 0 < \beta_0 < \beta_1, \alpha \in [\alpha_0, \alpha_1]$ and $\beta_0 \leq \beta \leq \beta_1$.

(a) For $\theta \in H^s$, the problem (4.7) has a unique solution

$$v_{\alpha, \beta, \theta} \in C([0, T]; H^s) \cap C_\alpha(T; H^{\beta+s}), \partial_t^\alpha v_{\alpha, \beta, \theta} \in C_\alpha(T, H^s)$$

which has the form

$$v(t) := v_{\alpha, \beta, \theta, f}(t) = \sum_{p=1}^{\infty} F_{\alpha, \beta, \theta, p}(t) \phi_p \quad (4.9)$$

where

$$F_{\alpha, \beta, \theta, k}(t) = \langle \theta, \phi_k \rangle E_{\alpha, 1}(-\lambda_k^\beta t^\alpha).$$

Moreover, there exists a constant C independent of α, β, θ such that

$$\|v\|_{C([0, T], H^s)} + \|v\|_{C_\alpha(T, H^{\beta+s})} + \|\partial_t^\alpha v\|_{C_\alpha(T, H^s)} \leq C \|\theta\|_{H^s}.$$

If, in addition, $s \geq \beta$ then

$$\partial_t^\alpha v_{\alpha, \beta, \theta} \in C([0, T], H^{s-\beta})$$

and

$$\|\partial_t^\alpha v\|_{C([0, T], H^{s-\beta})}^2 \leq C \|\theta\|_{H^s}^2.$$

(b) Let $s \geq 0, \alpha, \alpha' \in (0, 1), \beta' \in [\beta_0, \beta_1], \theta, \theta' \in H^s$.

(i) If $\theta' \rightarrow \theta$ in $H^s, \alpha' \rightarrow \alpha, \beta' \rightarrow \beta$ in \mathbb{R} then

$$\|v_{\alpha', \beta', \theta'} - v_{\alpha, \beta, \theta}\|_{H^s} \rightarrow 0.$$

(ii) If $\theta, \theta' \in H^s, s > \rho \geq 0, \alpha' \in [\alpha_0, \alpha_1], \beta' \in [\beta_0, \beta_1]$ then there exists a constant $C = C(\alpha_0, \alpha_1, \beta_0, \beta_1, s, \rho)$ such that

$$\|v_{\alpha', \beta', \theta'}(\cdot, t) - v_{\alpha, \beta, \theta}(\cdot, t)\|_{H^\rho}^2 \leq C \|\theta' - \theta\|_{H^s}^2 + C \|\theta\|_{H^s}^2 (|\alpha' - \alpha| + |\beta' - \beta|)^{2\gamma},$$

where $\gamma = \min\{1, (s - \rho)/\beta_1\}$.

Remark 4.1. For a large class of operators A , Chen et al. [8] and Meerschaert et al. [23] showed that the solution to equation (4.7) when $\beta = 1$, can be also represented as follows

$$\begin{aligned} u(t, x) &= \sum_{p=1}^{\infty} E_{\alpha, 1}(-\lambda_p t^\alpha) \langle \theta, \phi_p \rangle \phi_p(x) \\ &= \mathbb{E}_x[\theta(X(E_t)); \tau_\Omega(X) > E_t] \\ &= \frac{t}{\alpha} \int_0^\infty T_\Omega(u)(\theta(x)) g_\alpha(tl^{-1/\alpha}) u^{-1/\alpha-1} du = \int_0^\infty T_\Omega((t/u)^\alpha)(\theta(x)) g_\alpha(l) dl. \end{aligned}$$

where X is a process such that $v(t, x) = \mathbb{E}_x(\theta(X(t)), \tau_\Omega > t) = T_\Omega(t)(\theta(x))$ solves the equation (4.7) when $\beta = 1$, $\tau_\Omega = \inf\{s > 0 : X(s) \notin \Omega\}$ is the first exit time of X from Ω , g_α is the density of a stable subordinator Y_t of index $\alpha \in (0, 1)$ with the Laplace transform $E(e^{-sY_t}) = e^{-ts^\alpha}$, and $E_t = \inf\{\tau > 0 : Y(\tau) > t\}$ is the inverse of Y .

Remark 4.2. In Theorem 4.3 when $\beta = 1$, The operators A include Laplacian Δ in a bounded domain Ω in a Euclidean space \mathbb{R}^d with Dirichlet boundary condition, and fractional Laplacian $-(-\Delta)^\gamma$, $\gamma \in (0, 1)$ with exterior Dirichlet boundary conditions. The process mentioned in previous remark corresponding to the Laplacian is the killed Brownian motion. The process that corresponds to fractional Laplacian is a symmetric stable process. This fractional Laplacian is different operator than the operator one gets by taking the powers of the Laplacian with Dirichlet boundary condition using the spectral theory that was mentioned in equation (4.5), see Chen et al. [8] for more details. The most important differences are the facts that the eigenvalues are not powers of the eigenvalues of the Laplacian, and the eigenfunctions of the fractional Laplacian are not the same as the eigenfunctions of the Laplacian.

Proof of Theorem 4.3. We first prove the existence of solution of the problem (4.6) by using the Galerkin method combined with the spectral method as in [30]. Let $V_n = \text{span}\{\phi_1, \dots, \phi_n\}$, $\theta_n = \sum_{p=1}^n \langle \theta, \phi_p \rangle \phi_p$. We consider the problem of finding $v_n \in C([0, T]; V_n)$ such that

$$\frac{d}{dt} \langle J^{1-\alpha}(v_n - \theta_n, \phi) + a_\beta(v_n, \phi) = 0, \quad \forall \phi \in V_n.$$

Put

$$\begin{aligned} v_n(t) &:= v_{\alpha, \beta, \theta, n}(t) = \sum_{p=1}^n F_{\alpha, \beta, \theta, p}(t) \phi_p, \\ z_n(t) &:= z_{\alpha, \beta, \theta, n}(t) = - \sum_{p=1}^n \lambda_p^\beta c_p(t) \phi_p \end{aligned}$$

and choosing $\phi = \phi_k$, $k = \overline{1, n}$ we will obtain the fractional differential equation

$$\frac{d}{dt} J^{1-\alpha}(c_{nk} - \langle \theta, \phi_k \rangle) + \lambda_k^\beta c_{nk} = 0$$

which gives

$$c_{nk}(t) = F_{\alpha, \beta, \theta, k}(t).$$

Since c_{nk} is independent of n , we will write $c_{nk} = c_k$ and we obtain

Proof of (a): For $0 \leq \rho \leq \beta$, direct computation gives

$$\lambda_p^{2(\rho+s)} |F_{\alpha, \beta, \theta, p}(t)|^2 \leq C \frac{\lambda_p^{2(\rho+s)} |\langle \theta, \phi_p \rangle|^2}{(1 + \lambda_p^\beta t^\alpha)^2}.$$

Hence, for $\rho = 0$, we obtain

$$\lambda_p^{2s} |F_{\alpha, \beta, \theta, p}(t)|^2 \leq C \lambda_p^{2s} |\langle \theta, \phi_p \rangle|^2 \quad \text{and} \quad \sum_{p=1}^{\infty} \lambda_p^{2s} |\langle \theta, \phi_p \rangle|^2 = \|\theta\|_{H^s}^2 < \infty.$$

Hence we deduce that $\{v_n\}$ is uniformly convergent to the function

$$v(t) = \sum_{p=1}^{\infty} F_{\alpha, \beta, \theta, p}(t) \phi_p$$

in $C([0, T]; H^s)$ and that

$$\|v(t)\|_{H^s}^2 \leq C \sum_{p=1}^{\infty} \lambda_p^{2s} |\langle \theta, \phi_p \rangle|^2 = C \|\theta\|_{H^s}^2.$$

Choosing $\rho = \beta$, we obtain

$$\begin{aligned}\lambda_p^{2(\beta+s)}|F_{\alpha,\beta,\theta,p}(t)|^2 &\leq C \frac{\lambda_p^{2(\beta+s)}|\langle\theta, \phi_p\rangle|^2}{(1 + \lambda_p^\beta t^\alpha)^2}, \\ \frac{t^{2\alpha}\lambda_p^{2(\beta+s)}|\langle\theta, \phi_p\rangle|^2}{(1 + \lambda_p^\beta t^\alpha)^2} &\leq \sum_{p=1}^{\infty} \lambda_p^{2s}|\langle\theta, \phi_p\rangle|^2 = C\|\theta\|_{H^s}^2.\end{aligned}$$

We deduce that $\{v_n\}$ is uniformly convergent to the function v in $C_\alpha(T, H^{\beta+s})$ and

$$t^{2\alpha}\|v(t)\|_{H^{\beta+s}}^2 \leq \sum_{p=1}^{\infty} C \frac{t^{2\alpha}\lambda_p^{2(\beta+s)}|\langle\theta, \phi_p\rangle|^2}{(1 + \lambda_p^\beta t^\alpha)^2} \leq C \sum_{p=1}^{\infty} \lambda_p^{2s}|\langle\theta, \phi_p\rangle|^2 = C\|\theta\|_{H^s}^2.$$

We deduce that $\{v_n\}$ is uniformly convergent to the function v in $C([0, T]; H^s) \cap C_\alpha(T, H^{\beta+s})$. Combining these inequalities, we obtain

$$\|v\|_{C([0, T]; H^s)}^2 + \|v\|_{C_\alpha(T; H^{\beta+s})}^2 \leq C\|\theta\|_{H^s}^2.$$

To estimate $\partial_t^\alpha v_n$ we have

$$\lambda_p^{2s}|\lambda_p^\beta c_p(t)|^2 = \lambda_p^{2(\beta+s)}|F_{\alpha,\beta,\theta,p}(t)|^2$$

and

$$t^{2\alpha} \sum_{p=1}^{\infty} \lambda_p^{2(\beta+s)}|F_{\alpha,\beta,\theta,p}(t)|^2 \leq C \sum_{p=1}^{\infty} \frac{t^{2\alpha}\lambda_p^{2(\beta+s)}|\langle\theta, \phi_p\rangle|^2}{(1 + \lambda_p^\beta t^\alpha)^2} \leq C \sum_{p=1}^{\infty} \lambda_p^{2s}|\langle\theta, \phi_p\rangle|^2 = \|\theta\|_{H^s}^2 < \infty.$$

Hence, z_n converge uniformly to a function z in $C_\alpha(T, H^s)$ and

$$\|z\|_{C_\alpha(T, H^s)}^2 \leq C\|\theta\|_{H^s}^2.$$

We can verify as in Theorem 4.2 that v is a solution of the problem (4.7). Moreover, since $z_n = \partial_t^\alpha v_n$, we can obtain by integration by parts

$$\int_0^T \varphi(t) z_n(t) dt = - \int_0^T \varphi'(t) J^{1-\alpha} v_n(t) dt \quad \text{for all } \varphi \in C_c^\infty(0, T).$$

Let $n \rightarrow \infty$ gives

$$\int_0^T \varphi(t) z(t) dt = - \int_0^T \varphi'(t) J^{1-\alpha} v(t) dt \quad \text{for all } \varphi \in C_c^\infty(0, T).$$

Hence $z = \partial_t^\alpha v$. Combining the estimates for $v, \partial_t^\alpha v$ gives

$$\|v\|_{C([0, T], H^s)} + \|u\|_{C_\alpha(T, H^{\beta+s})} + \|\partial_t^\alpha u\|_{C_\alpha(T, H^s)} \leq C\|\theta\|_{H^s}.$$

Now, if $s \geq \beta$ and $f \in C([0, T]; H^r)$ then

$$\lambda_p^{2(s-\beta)}|\lambda_p^\beta c_p(t)|^2 = \lambda_p^{2(s-\beta)}\lambda_p^{2\beta}|F_{\alpha,\beta,\theta,p}(t)|^2 \leq \lambda_p^{2(s-\beta)}\lambda_p^{2\beta}|\langle\theta, \phi_p\rangle|^2 = C\lambda_p^{2s}|\langle\theta, \phi_p\rangle|^2,$$

and

$$\sum_{p=1}^{\infty} C\lambda_p^{2s}|\langle\theta, \phi_p\rangle|^2 = C\|\theta\|_{H^s}^2 \leq \infty.$$

Hence z_n converges uniformly to the function $\partial_t^\alpha v$ in $C([0, T]; H^{s-\beta})$ and

$$\|\partial_t^\alpha v\|_{C([0, T]; H^{s-\beta})}^2 \leq C\|\theta\|_{H^s}^2.$$

Proof of (b): In the rest of the proof we will consider the continuity of the solution with respect to the parameter α, β , the initial condition θ . From Part (b) of Lemma 3.3, we obtain

$$\left| E_{\alpha,1}(-\lambda_k^\beta t) - E_{\alpha',1}(-\lambda_k^{\beta'} t) \right| \leq C\lambda_k^{2\beta_1} (|\beta' - \beta| + |\alpha' - \alpha|).$$

Recall that we have

$$v_{\alpha,\beta,\theta}(t) = \sum_{p=1}^{\infty} E_{\alpha,1}(-\lambda_p^\beta t) \langle \theta, \varphi_p \rangle \varphi_p.$$

So, for $\rho > 0$ we obtain

$$\begin{aligned} \|v_{\alpha,\beta,\theta}(t) - v_{\alpha',\beta',\theta'}(t)\|_{H^\rho}^2 &\leq 2 \sum_{p=1}^{\infty} \lambda_p^{2\rho} \left[E_{\alpha,1}(-\lambda_p^\beta t) - E_{\alpha',1}(-\lambda_p^{\beta'} t) \right]^2 \langle \theta, \varphi_p \rangle^2 \\ &\quad + 2 \sum_{p=1}^{\infty} \lambda_p^{2\rho} E_{\alpha',1}^2(-\lambda_p^{\beta'} t) \langle \theta' - \theta, \varphi_p \rangle^2. \end{aligned}$$

We separate the first sum into two sums, one sum is from $p = 1$ to $p = N$ and one sum is from $p = N + 1$ to infinity. Using the fact that $0 \leq E_{\alpha,1}(z) \leq 1$ for $z \leq 0$ and Part (b) of Lemma 3.3 we obtain

$$\begin{aligned} \|v_{\alpha',\beta',\theta'}(t) - v_{\alpha,\beta,\theta}(t)\|_{H^\rho}^2 &\leq C(|\alpha' - \alpha| + |\beta' - \beta|)^2 \sum_{p=1}^N \lambda_p^{2(\beta_1+\rho)} \langle \theta, \varphi_p \rangle^2 + \\ &\quad 2 \sum_{p=N+1}^{\infty} \lambda_p^{2\rho} \langle \theta, \varphi_p \rangle^2 + 2\|\theta' - \theta\|_{H^\rho}^2. \end{aligned}$$

(i) We choose a sequence $(\alpha_n, \beta_n, \theta_n) \in (0, 1) \times [\beta_0, \beta_1] \times H$ such that $(\alpha_n, \beta_n, \theta_n) \rightarrow (\alpha, \beta, \theta)$. It follows that

$$\limsup_{n \rightarrow \infty} \|v_{\alpha,\beta,\theta}(t) - v_{\alpha_n,\beta_n,\theta_n}(t)\|_{H^s}^2 \leq 2 \sum_{p=N+1}^{\infty} \lambda_p^{2s} \langle \theta, \varphi_p \rangle^2.$$

Letting $N \rightarrow \infty$ gives

$$\limsup_{n \rightarrow \infty} \|v_{\alpha,\beta,\theta}(t) - v_{\alpha_n,\beta_n,\theta_n}(t)\|_{H^s}^2 = 0.$$

Hence, we obtain (i).

(ii) Now, if $\theta \in H^s$, $s > \beta_1 + \rho$ then

$$\begin{aligned} \|v_{\alpha',\beta',\theta'}(t) - v_{\alpha,\beta,\theta}(t)\|_{H^\rho}^2 &\leq C(|\alpha' - \alpha| + |\beta' - \beta|)^2 \sum_{p=1}^{\infty} \lambda_p^{2(\beta_1+\rho)} \langle \theta, \varphi_p \rangle^2 + 2\|\theta' - \theta\|_{H^\rho}^2 \\ &\leq C(|\alpha' - \alpha| + |\beta' - \beta|)^2 \sum_{p=1}^{\infty} \lambda_p^{2s} \langle \theta, \varphi_p \rangle^2 + 2\|\theta' - \theta\|_{H^s}^2 \\ &\leq C(|\alpha' - \alpha| + |\beta' - \beta|)^2 \|\theta\|_{H^s}^2 + 2\|\theta' - \theta\|_{H^s}^2. \end{aligned}$$

If $\theta \in H^s$, $0 < s \leq \beta_1 + \rho$ then

$$\begin{aligned}
\|v_{\alpha', \beta', \theta'}(t) - v_{\alpha, \beta, \theta}(t)\|_{H^\rho}^2 &\leq C(|\alpha' - \alpha| + |\beta' - \beta|)^2 \sum_{p=1}^N \lambda_p^{2(\beta_1 + \rho)} \langle \theta, \varphi_p \rangle^2 + \\
&\quad 2 \sum_{p=N+1}^{\infty} \lambda_p^{2\rho} \langle \theta, \varphi_p \rangle^2 + 2\|\theta' - \theta\|_{H^\rho}^2 \\
&\leq C(|\alpha' - \alpha| + |\beta' - \beta|)^2 \lambda_N^{2(\beta_1 + \rho - s)} \sum_{p=1}^N \lambda_p^{2s} \langle \theta, \varphi_p \rangle^2 + \\
&\quad 2\lambda_{N+1}^{-2(s-\rho)} \sum_{p=N+1}^{\infty} \lambda_p^{2s} \langle \theta, \varphi_p \rangle^2 + 2\|\theta' - \theta\|^2 \\
&\leq C \left((|\alpha' - \alpha| + |\beta' - \beta|)^2 \lambda_N^{2(\beta_1 + \rho - s)} + 2\lambda_{N+1}^{-2(s-\rho)} \right) \|\theta\|_{H^s}^2 + 2\|\theta' - \theta\|_{H^s}^2.
\end{aligned}$$

Choose N such that $\lambda_{N+1}^{-1} \leq (|\alpha' - \alpha| + |\beta' - \beta|)^{1/\beta_1} \leq \lambda_N^{-1}$ gives

$$\|v_{\alpha', \beta', \theta'}(t) - v_{\alpha, \beta, \theta}(t)\|_{H^\rho}^2 \leq C(|\alpha' - \alpha| + |\beta' - \beta|)^{2(s-\rho)/\beta_1} \|\theta\|_{H^s}^2 + 2\|\theta' - \theta\|_{H^s}^2.$$

□

Next we establish results for the non-homogeneous problem (4.8) with zero initial value.

Theorem 4.4. *Let $r \geq 0$, $0 < \alpha_0 < \alpha_1 < 1$, $0 < \beta_0 < \beta_1$, $\alpha \in [\alpha_0, \alpha_1]$ and $\beta_0 \leq \beta \leq \beta_1$.*

(a) *For $f \in L^2(0, T; H^r)$, the problem (4.8) has a unique solution*

$$w_{\alpha, \beta, f} \in C([0, T]; H^{\beta+r}), \partial_t^\alpha w_{\alpha, \beta, f} \in L^2(0, T; H^r)$$

which has the form

$$w(t) := w_{\alpha, \beta, f}(t) = \sum_{p=1}^{\infty} G_{\alpha, \beta, f, p}(t) \phi_p \quad (4.10)$$

where

$$G_{\alpha, \beta, f, k}(t) = \int_0^t \langle f(\tau), \phi_k \rangle (t - \tau)^{\alpha-1} E_{\alpha, \alpha}(-\lambda_k^\beta (t - \tau)^\alpha) d\tau.$$

Moreover, there exists a constant C independent of α, β, f such that

$$\|w\|_{C([0, T], H^{\beta+r})} + \|\partial_t^\alpha w\|_{L_{\alpha, r}^2(T)} \leq C \|f\|_{L^2(0, T; H^r)}.$$

If, in addition, $f \in C([0, T], H^r)$ then

$$\partial_t^\alpha w_{\alpha, \beta, f} \in C([0, T], H^r)$$

and

$$\|\partial_t^\alpha w\|_{C([0, T]; H^r)}^2 \leq C \|f\|_{C([0, T]; H^r)}^2.$$

(b) For $f \in L^2(0, T; H^r)$, we have

$$w_{\alpha', \beta', f'} \rightarrow w_{\alpha, \beta, f} \quad \text{in } L^2(0, T; H^{\beta_0 + r})$$

as $(\alpha', \beta', f') \rightarrow (\alpha, \beta, f)$ in $\mathbb{R} \times \mathbb{R} \times L^2(0, T; H^r)$ and

$$\begin{aligned} \|w_{\alpha', \beta', f'} - w_{\alpha, \beta, f}\|_{L^2(0, T; H^{\rho+r})}^2 &\leq C_\delta (|\alpha - \alpha'| + |\beta - \beta'|)^{2(\beta_0 - \rho)\mu} \|f\|_{L^2(0, T; H^r)}^2 + \\ &\quad C \|f' - f\|_{L^2(0, T; H^r)}^2 \end{aligned}$$

for $\delta > 0, 0 < \rho < \beta_0$ and $\mu = (\beta_0 + \beta_1 + \delta)^{-1}$.

Proof. Put $V_n = \text{span}\{\phi_1, \dots, \phi_n\}$ as in the proof of the previous theorem. We consider the problem of finding $w_n \in C([0, T]; V_n)$ such that

$$\frac{d}{dt} \langle J^{1-\alpha} w_n, v \rangle + a_\beta(w_n, v) = \langle f(t), v \rangle, \quad \forall v \in V_n.$$

By putting $w_n(t) = w_{\alpha, \beta, f, n}(t) := \sum_{p=1}^n d_{np}(t) \phi_p$ and choosing $v = \phi_k, k = \overline{1, n}$ we will obtain the fractional differential equation

$$\frac{d}{dt} J^{1-\alpha} d_{nk} + \lambda_k^\beta d_{nk} = \langle f(t), \phi_k \rangle$$

which gives

$$d_{nk}(t) = G_{\alpha, \beta, f, k}(t).$$

Since d_{nk} is independent of n , we will denote $d_{nk} = d_k$. We denote

$$\begin{aligned} w_n(t) &:= w_{\alpha, \beta, f, n}(t) = \sum_{p=1}^n G_{\alpha, \beta, f, p}(t) \phi_p, \\ h_n(t) &:= h_{\alpha, \beta, f, n}(t) = \sum_{p=1}^n (-\lambda_p^\beta d_p(t) + \langle f(t), \phi_p \rangle) \phi_p. \end{aligned}$$

Proof of (a): We estimate the Fourier coefficients of the function w_n . We first have

$$\begin{aligned} |G_{\alpha, \beta, f, p}(t)|^2 &\leq \int_0^t |\langle f(\tau), \phi_p \rangle|^2 d\tau \left(\int_0^t \tau^{\alpha-1} E_{\alpha, \alpha}(-\lambda_p^\beta \tau^\alpha) d\tau \right)^2 \\ &= \frac{1}{\lambda_p^{2\beta}} \int_0^t |\langle f(\tau), \phi_p \rangle|^2 d\tau \left(\int_0^t \frac{d}{d\tau} E_{\alpha, 1}(-\lambda_p^\beta \tau^\alpha) d\tau \right)^2 \\ &= \frac{(1 - E_{\alpha, 1}(-\lambda_p^\beta t^\alpha))^2}{\lambda_p^{2\beta}} \int_0^t |\langle f(\tau), \phi_p \rangle|^2 d\tau \\ &\leq \frac{1}{\lambda_p^{2\beta}} \int_0^T |\langle f(\tau), \phi_p \rangle|^2 d\tau. \end{aligned}$$

Hence, we have

$$\begin{aligned} \lambda_p^{2(\beta+r)} |G_{\alpha, \beta, f, p}(t)|^2 &\leq \lambda_p^{2r} \int_0^T |\langle f(\tau), \phi_p \rangle|^2 d\tau \\ \sum_{p=1}^\infty \lambda_p^{2(\beta+r)} |G_{\alpha, \beta, f, p}(t)|^2 &\leq \sum_{p=1}^\infty \lambda_p^{2r} \int_0^T |\langle f(\tau), \phi_p \rangle|^2 d\tau = C \|f\|_{L^2(0, T; H^r)}^2. \end{aligned}$$

This implies that $\{w_n\}$ is uniformly convergent to the function

$$w(t) := w_{\alpha,\beta,f}(t) = \sum_{p=1}^{\infty} G_{\alpha,\beta,f,p}(t) \phi_p$$

in $C([0, T]; H^{\beta+r})$ and

$$\|w(t)\|_{H^{\beta+r}}^2 = \sum_{p=1}^{\infty} \lambda_p^{2(\beta+r)} |G_{\alpha,\beta,f,p}(t)|^2 \leq \sum_{p=1}^{\infty} \lambda_p^{2r} \int_0^T |\langle f(\tau), \phi_p \rangle|^2 d\tau = C \|f\|_{L^2(0,T;H^r)}^2.$$

We can verify as in Theorem 4.2 that w is a solution of the problem (4.8). Now, we have

$$\lambda_p^{2r} |\lambda_p^\beta d_p(t) + \langle f(t), \phi_p \rangle|^2 \leq 2\lambda_p^{2(\beta+r)} |G_{\alpha,\beta,f,p}(t)|^2 + 2\lambda_p^{2r} |\langle f(t), \phi_p \rangle|^2,$$

and

$$\int_0^T \sum_{p=1}^{\infty} \left\{ 2\lambda_p^{2(\beta+r)} |G_{\alpha,\beta,f,p}(t)|^2 + 2\lambda_p^{2r} |\langle f(t), \phi_p \rangle|^2 \right\} dt \leq C \|f\|_{L^2(0,T;H^r)}^2.$$

Hence, h_n converge uniformly to a function h in $L^2(0, T; H^r)$ and

$$\|h\|_{L^2(0,T;H^r)}^2 \leq C \|f\|_{L^2(0,T;H^r)}^2.$$

Moreover, since $h_n = \partial_t^\alpha w_n$, we can prove as in the proof of the previous theorem that $h = \partial_t^\alpha u$. Combining the estimates for $w, \partial_t^\alpha w$ gives

$$\|w\|_{C([0,T];H^{\beta+r})} + \|\partial_t^\alpha w\|_{L^2(0,T;H^r)} \leq C \|f\|_{L^2(0,T;H^r)}.$$

Now, if $f \in C([0, T]; H^r)$ then

$$\sum_{p=1}^{\infty} \left\{ 2\lambda_p^{2(\beta+r)} |G_{\alpha,\beta,f,p}(t)|^2 + 2\lambda_p^{2r} |\langle f(t), \phi_p \rangle|^2 \right\} \leq C \|f\|_{C([0,T];H^r)}^2.$$

Hence h_n converges uniformly to the function $\partial_t^\alpha w$ in $C([0, T]; H^r)$ and

$$\|\partial_t^\alpha w\|_{C([0,T];H^r)}^2 \leq C \|f\|_{C([0,T];H^r)}^2.$$

Proof of (b): Finally, for $0 < \rho \leq \beta_0$ we estimate

$$\begin{aligned} \|w_{\alpha',\beta',f'} - w_{\alpha,\beta,f}\|_{L^2(0,T;H^{\rho+r})}^2 &= \sum_{p=1}^{\infty} \lambda_p^{\rho+r} |G_{\alpha',\beta',f',p}(t) - G_{\alpha,\beta,f,p}(t)|^2 \\ &\leq 2 \sum_{p=1}^{\infty} \lambda_p^{\rho+r} |G_{\alpha',\beta',f',p}(t) - G_{\alpha',\beta',f,p}(t)|^2 \\ &\quad + 2 \sum_{p=1}^{\infty} \lambda_p^{\rho+r} |G_{\alpha',\beta',f,p}(t) - G_{\alpha,\beta,f,p}(t)|^2. \end{aligned}$$

For the first term, we have

$$\begin{aligned} \sum_{p=1}^{\infty} \lambda_p^{\rho+r} |G_{\alpha',\beta',f',p}(t) - G_{\alpha',\beta',f,p}(t)|^2 &= 2 \sum_{p=1}^{\infty} \lambda_p^{\rho+r} |G_{\alpha',\beta',f'-f,p}(t)|^2 \\ &\leq C \sum_{p=1}^{\infty} \lambda_p^{\rho-\beta'} \lambda_p^r |\langle f' - f, \phi_p \rangle|^2 \leq C \|f' - f\|_{L^2(0,T;H^r)}^2. \end{aligned}$$

We consider the second term. Choose an $N \in \mathbb{N}$. From Lemma 3.3 part (c) we obtain

$$\begin{aligned}
\|w_{\alpha',\beta',f} - w_{\alpha,\beta,f}\|_{L^2(0,T;H^{\rho+r})}^2 &\leq C \sum_{p=1}^N \lambda_p^{2(\rho+r)} \left[|\alpha - \alpha'| (1 + |\lambda_p|^\beta) + |\lambda_p^{\beta'} - \lambda_p^\beta| \right]^2 \int_0^T |\langle f(\cdot), \phi_p \rangle|^2 dt \\
&\quad + 2 \sum_{p=N+1}^{\infty} \lambda_p^{2(\rho+r)} \int_0^T \{G_{\alpha',\beta',f',p}^2(t) + G_{\alpha,\beta,f,p}^2(t)\} dt \\
&\leq C \sum_{p=1}^N \lambda_p^{2(\rho+r)} \left[|\alpha - \alpha'| (1 + |\lambda_p|^\beta) + |(\beta - \beta') \lambda_p^{\beta_1} \ln \lambda_p| \right]^2 \int_0^T |\langle f(t), \phi_p \rangle|^2 dt \\
&\quad + C \sum_{p=N+1}^{\infty} \lambda_p^{2(\rho+r)} \left\{ \frac{1}{\lambda_p^{2\beta'}} + \frac{1}{\lambda_p^{2\beta}} \right\} \int_0^T |\langle f(t), \phi_p \rangle|^2 dt \\
&\leq C \lambda_N^{2(\rho+\beta_1)} |\ln \lambda_N|^2 (|\alpha - \alpha'| + |\beta - \beta'|)^2 \|f\|_{L^2(0,T;H^r)}^2 \\
&\quad + \lambda_{N+1}^{2(\rho-\beta_0)} \sum_{p=N+1}^{\infty} \int_0^T \lambda_p^r |\langle f(t), \phi_p \rangle|^2 dt.
\end{aligned}$$

Now, for $\rho = \beta_0$, we can verify directly that

$$w_{\alpha',\beta,f'} \rightarrow w_{\alpha,\beta,f} \quad \text{in } L^2(0,T;H^{\beta_0+r})$$

as $(\alpha', \beta', f') \rightarrow (\alpha, \beta, f)$ in $\mathbb{R} \times \mathbb{R} \times L^2(0,T;H^r)$. For $0 \leq \rho < \beta_0$, we choose N such that

$$\lambda_N \leq (|\alpha - \alpha'| + |\beta - \beta'|)^{-\mu} \leq \lambda_{N+1}$$

where μ satisfies $1 - \mu(\rho + \beta_1 + \delta) = \mu(\beta_0 - \rho)$, i.e., $\mu = (\beta_0 + \beta_1 + \delta)^{-1}$. Then we have

$$\begin{aligned}
\|w_{\alpha',\beta',f} - w_{\alpha,\beta,f}\|_{L^2(0,T;H^{\rho+r})}^2 &\leq C_\delta (|\alpha - \alpha'| + |\beta - \beta'|)^{2-2\mu(\rho+\beta_1+\delta)} \|f\|_{L^2(0,T;H^r)}^2 + \\
&\quad C (|\alpha - \alpha'| + |\beta - \beta'|)^{2(\beta_0-\rho)\mu} \|f\|_{L^2(0,T;H^r)}^2 \\
&\leq (C_\delta + C) (|\alpha - \alpha'| + |\beta - \beta'|)^{2(\beta_0-\rho)\mu} \|f\|_{L^2(0,T;H^r)}^2.
\end{aligned}$$

It follows that

$$\|w_{\alpha',\beta',f'} - w_{\alpha,\beta,f}\|_{L^2(0,T;H^{\rho+r})}^2 \leq C_\delta (|\alpha - \alpha'| + |\beta - \beta'|)^{2(\beta_0-\rho)\mu} \|f\|_{L^2(0,T;H^r)}^2 + C \|f' - f\|_{L^2(0,T;H^r)}^2.$$

□

5 Instability of the solutions of some Ill-posed problems

In this section, we will give some definitions and examples for showing the instability of solutions in the case of the fractional order is noised. First, we introduce general theory of stability and instability of inverse problems which depended on the (noise) fractional order. Next, we present some examples for this general theory.

5.1 General theory

Let X, Y be two Banach spaces, $K : X \rightarrow Y$. In literature of inverse problems, we often use Hadamard's definition of ill-posedness for the problem $Kx = y$. Now, we first consider the

instability of inverse problems which depended on the (noise) fractional order α . To make the situation clear, we develop a concept of instability for a family of operator K_β upon the generic parametric β and discuss some examples. Let $\beta_0 \in \mathbb{R}$, $x_0, u_0 \in X$, $y_0, v_0 \in Y$ and $[a, b]$ is an interval in \mathbb{R} such that $\beta_0 \in [a, b]$. We consider the family of (linear or nonlinear) operators $K_\beta : X \rightarrow Y$ where $\beta \in [a, b]$. Assume that $K_{\beta_0}x_0 = y_0$, $Ku_0 = v_0$.

Definition 5.1. *We say that*

- (a) *the operator K has an unstable inverse at u_0 if there exist sequences $(u_n) \subset X$ such that $Ku_n \rightarrow v_0$ but $u_n \not\rightarrow u_0$ as $n \rightarrow \infty$.*
- (b) *the family (K_β) has a β_0 -unstable inverse at x_0 if there exist sequences $(\beta_n) \subset (a, b)$, $(x_n) \subset X$ such that $\beta_n \rightarrow \beta_0$, $K_{\beta_n}x_n \rightarrow y_0$ but $x_n \not\rightarrow x_0$ as $n \rightarrow \infty$.*
- (c) *the family (K_β) has a properly β_0 -unstable inverse at x_0 if there exist sequences $(\beta_n) \subset (a, b)$, $(x_n) \subset X$, $(y_n) \subset Y$ such that $K_{\beta_n}x_n = y_n$, $K_{\beta_0}x_n^* = y_n$ and*

$$\beta_n \rightarrow \beta_0, y_n \rightarrow y_0, x_n^* \rightarrow x_0 \text{ but } x_n \not\rightarrow x_0 \text{ as } n \rightarrow \infty.$$
- (d) *the operator $R_\delta : Y \rightarrow X$ is a regularization at β_0 of the family (K_β) if*

$$\lim_{(\delta, \beta) \rightarrow (0, \beta_0)} R_\delta K_\beta x = x \quad \text{for } x \in X.$$

We have

Theorem 5.1. *With the notations in the definition above, we obtain the following results.*

- (a) *If $K_\beta x \rightarrow K_{\beta_0}x$ as $\beta \rightarrow \beta_0$ for every $x \in X$ and K_{β_0} has an unstable inverse at x_0 then (K_β) has a β_0 -unstable inverse at x_0 .*
- (b) *If $\sup_{\|x\|_X \leq M} \|K_\beta x - K_{\beta_0}x\| \rightarrow 0$ as $\beta \rightarrow \beta_0$ and (K_β) has the β_0 -unstable inverse at x_0 with $\sup_n \|x_n\|_X < \infty$ then K_{β_0} has the unstable inverse at x_0 .*
- (c) *If K_β is bounded linear and that $\lim_{\beta \rightarrow \beta_0} \|K_\beta - K_{\beta_0}\| = 0$ and (K_β) has the β_0 -unstable inverse at x_0 then K_{β_0} has the unstable inverse at x_0 .*

Proof. Proof of (a): Assume that $K_{\beta_0}(u_n) = v_n$, $v_n \rightarrow v_0$ in Y , $u_n \not\rightarrow u_0$ in X . For each $n \in \mathbb{N}$, we choose $\beta_n \in (a, b)$ such that

$$|\beta_n - \beta_0| + \|K_{\beta_n}u_n - K_{\beta_0}u_n\| \leq \frac{1}{n} \text{ as } n \rightarrow \infty.$$

Put $y_n = K_{\beta_n}u_n$, $x_n = u_n$ we obtain that (K_β) has the β_0 -unstable inverse at x_0 .

Proof of (b): Assume that $K_{\beta_n}x_n = y_n$, $y_n \rightarrow y_0$, $x_n \not\rightarrow x_0$ and $\sup_n \|x_n\|_X \leq M$. We have

$$\lim_{n \rightarrow \infty} \|K_{\beta_0}x_n - K_{\beta_n}x_n\|_Y = 0.$$

It follows that $K_{\beta_0}x_n \rightarrow y_0$ and $x_n \not\rightarrow x_0$ as $n \rightarrow \infty$. Hence K_{β_0} has the unstable inverse at x_0 .

Proof of (c): Assume that $K_{\beta_n}x_n = y_n$, $y_n \rightarrow y_0$, $x_n \not\rightarrow x_0$. If $\sup_n \|x_n\|_X \leq M$ we can use (b) to prove the result. If $\sup_n \|x_n\|_X = \infty$, we can choose a subsequence x_{n_k} such that $\lim_{k \rightarrow \infty} \|x_{n_k}\|_X = \infty$. Putting $\tilde{x}_k = x_0 + z_k$ with $z_k = \frac{x_{n_k}}{\|x_{n_k}\|_X}$ we have $\|\tilde{x}_k - x_0\|_X = \|z_k\|_X = 1$,

$$\lim_{k \rightarrow \infty} \|K_{\beta_{n_k}}z_k - K_{\beta_0}z_k\|_Y = 0, \quad \lim_{k \rightarrow \infty} \|K_{\beta_{n_k}}z_k\|_Y = \lim_{k \rightarrow \infty} \|y_{n_k}\|_Y / \|x_{n_k}\|_X = 0.$$

Hence $\lim_{k \rightarrow \infty} K_{\beta_0}z_k = 0$ and

$$K_{\beta_0}\tilde{x}_k = y_0 + K_{\beta_0}z_k \rightarrow y_0 \quad \text{as } k \rightarrow \infty.$$

It follows that K_{β_0} has the unstable inverse at x_0 . □

5.2 Some examples for instability of fractional order

5.2.1 The first example

In this subsection, we present an example to show that the β_0 -unstable inverse cannot imply the unstable inverse of K_{β_0} . For $0 < \alpha < 1$, we consider the Abel operators $J_{-\infty}^\alpha : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ defined by

$$J_{-\infty}^\alpha u(x) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^x (x-t)^{\alpha-1} u(t) dt.$$

From [16, page 96], we have $\lim_{\alpha \rightarrow 0} J_{-\infty}^\alpha u = u$ for every $u \in H^1(\mathbb{R})$. So we can define $J_{-\infty}^0 = Id$ and $(J_{-\infty}^0)^{-1} = Id$ is continuous. Now, we prove that the family $(J_{-\infty}^\alpha)$ has the 0-unstable inverse. If $J_{-\infty}^\alpha u(x) = f(x)$ then $\hat{u}(\tau) = (i\tau)^\alpha \hat{f}(\tau)$. For $a_n, \delta_n > 0$, we put

$$f_n(x) = \frac{1}{2\pi} \left(\int_{a_n}^{a_n+\delta_n} e^{itx} dt + \int_{-a_n-\delta_n}^{-a_n} e^{itx} dt \right).$$

We have $\hat{f}_n = \chi_{(a_n, a_n+\delta_n)} + \chi_{(-a_n-\delta_n, -a_n)}$. It follows that $\|f\|^2 = \frac{1}{2\pi} \|\hat{f}_n\|^2 = \frac{\delta_n}{\pi}$. On the other hand, we have $J_{-\infty}^{\alpha_n} u_n(x) = f_n(x)$ for $\hat{u}_n(\tau) = (i\tau)^{\alpha_n} (\chi_{(a_n, a_n+\delta_n)} + \chi_{(-a_n-\delta_n, -a_n)})$. So we have

$$\|u_n\|^2 = \frac{1}{2\pi} \|\hat{u}_n\|^2 = \frac{1}{\pi} \int_{a_n}^{a_n+\delta_n} |\tau|^{2\alpha_n} d\tau \geq \frac{1}{\pi} a_n^{2\alpha_n} \delta_n.$$

Now, if we choose $a_n = n^n, \alpha_n = \delta_n = \frac{1}{n}$ then $J_{-\infty}^{\alpha_n} u_n = f_n$ with $\alpha_n \rightarrow 0, f_n \rightarrow 0$ but $u_n \not\rightarrow 0$. Hence $(J_{-\infty}^\alpha)$ has the 0-unstable inverse.

5.2.2 The second example

We consider the problem of finding $u_f \in L^2(\mathbb{R})$ from the given exact data $f \in L^2(\mathbb{R})$ such that

$$\hat{u}_{\alpha, f}(\tau) = e^{a\tau^\alpha} \hat{f}(\tau)$$

where $a > 0$ is constant.

First, we consider the given data $\alpha > 0$ and $f_0 = 0$. Then $\hat{u}_{\alpha, f_0}(\tau) = 0$. Assume that (α, f_0) is noised by $(\alpha + \epsilon_n, f_n)$ where $f_n \in L^2(\mathbb{R})$ such that $\hat{f}_n = n\chi_{(n^n, n^n + \frac{1}{n^3})}$ and $\epsilon_n = \frac{1}{n}$. Since the equality

$$\|f_n\|_{L^2(\mathbb{R})} = \|\hat{f}_n\|_{L^2(\mathbb{R})} = \int_{n^n}^{n^n + \frac{1}{n^3}} n^2 d\tau = \frac{1}{n}$$

we know that

$$|\alpha + \epsilon_n - \alpha| + \|f_n - f_0\|_{L^2(\mathbb{R})} = \frac{2}{n} \rightarrow 0, \quad n \rightarrow \infty.$$

And we have

$$\hat{u}_{\alpha+\epsilon_n, f_n}(\tau) = e^{a\tau^{\alpha+\epsilon_n}} \hat{f}_n(\tau) = n\chi_{(n^n, n^n + \frac{1}{n^3})} e^{a\tau^{\alpha+\epsilon_n}}$$

The norm of $u_{\alpha+\epsilon_n, f_n}$ in L^2 is estimated as follows

$$\begin{aligned} \|\hat{u}_{\alpha+\epsilon_n, f_n} - u_{\alpha, f_0}\|_{L^2(\mathbb{R})} &= \|\hat{u}_{\alpha+\epsilon_n, f_n}\|_{L^2(\mathbb{R})} \\ &= \int_{n^n}^{n^n + \frac{1}{n^3}} n^2 e^{2a\tau^{\alpha+\epsilon_n}} \chi_{(n^n, n^n + \frac{1}{n^3})} d\tau \geq \frac{e^{2an}}{n} \rightarrow +\infty, \quad n \rightarrow +\infty \end{aligned}$$

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